

POLAR MANIFOLDS AND ACTIONS

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Dedicated to Richard Palais on his 80th birthday

Many geometrically interesting objects arise from spaces supporting large transformation groups. For example symmetric spaces form a rich and important class of spaces. More generally special geometries such as, e.g., Einstein manifolds, manifolds of positive curvature, etc., are often first found and possibly classified in the homogeneous case [Be, Wa, AW, BB, Wan, BWZ, La]. Similarly, interesting minimal submanifolds have been found and constructed as invariant submanifolds relative to a large isometry group (see, e.g., [HHS] and [HL]).

The “almost homogeneous” case, i.e., manifolds with an isometric G action with one dimensional orbit space, has also received a lot of attention recently. They are special in that one can reconstruct the manifold from the knowledge of its isotropy groups. This played a crucial role in the construction of a large class of new manifolds with nonnegative curvature in [GZ1, GZ2], in the classification problem for positively curved manifolds of cohomogeneity one in [V1, V2, GWZ, VZ], as well as in the construction of a new manifold of positive curvature [De, GVZ].

Our main objective in this paper is to exhibit and analyze G manifolds M , that like manifolds with orbit space an interval, can be reconstructed from its isotropy groups. Although such manifolds are rather rigid compared with all G manifolds, there are several natural constructions of such manifolds, and they form a rich and interesting class, including manifolds related to Coxeter groups and Tits buildings as observed and used in [FGT].

The special actions of interest to us here are the so-called *polar actions*, i.e., proper isometric actions for which there is an (immersed) submanifold $\sigma : \Sigma \rightarrow M$, a so-called *section*, that meets all orbits orthogonally, or equivalently *the horizontal distribution* (on the regular part) *is integrable*. This concept was pioneered by Szenthe in [Sz1, Sz2] and independently by Palais and Terng in [PT1]. In fact Palais and Terng observed that this is the natural class of group actions where the classical Chevalley restriction theorem for the adjoint action of a compact Lie group holds, i.e., a G invariant function on M is smooth if and only if its restriction to a section is smooth (and invariant under the stabilizer of Σ). This is also a natural class of group actions where a reduction to a potentially simpler lower dimensional problem along a smooth section is possible since in general the smooth structure of the quotient M/G is too complicated to do analysis effectively. This has already been used successfully in the cohomogeneity one case where solutions to a PDE were obtained via a reduction to an ODE, see, e.g., [Bö, Co, Cs, CGLP] where manifolds with special holonomy, Einstein metrics, Sasakian Einstein metrics and soliton metrics were produced.

The most basic examples of polar actions are the adjoint action of a compact Lie group on itself, or on its Lie algebra. More generally, the isotropy representation of a symmetric space M , either on M , or on $T_p M$, is polar. In fact Dadok showed that a linear representation which is polar is (up to orbit equivalence) the isotropy representation of a symmetric space [Da].

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Polar actions on symmetric spaces (isometric with respect to the symmetric metric) have been studied extensively, see, e.g., [HPTT1, HPTT2, HLO]. They were classified for compact rank one symmetric spaces in [PTh] and for compact irreducible symmetric spaces of higher rank it was shown [Ko2, Ko3, KL, L] that a polar action must be hyperpolar, i.e., the section is flat, and the hyperpolar actions were classified in [Ko1]. For non-compact symmetric actions the classification is still wide open, see, e.g., [BD, BDT].

To describe the data involved in the reconstruction of a polar G manifold M , let $\sigma : \Sigma \rightarrow M$ be a section and Π its stabilizer group (mod kernel), which we refer to as the *polar group*. Then Σ is a totally geodesic immersed submanifold of M and Π a discrete group of isometries acting properly discontinuously on Σ . Moreover $M/G = \Sigma/\Pi$. The polar group contains a canonical general type “*reflection group*” $R \triangleleft \Pi$ with “fundamental domain” a *chamber*, $C \subset \Sigma$, locally the union of convex subsets of Σ (in the atypical case where R is trivial C is all of Σ). Furthermore, we have the subgroup of the polar group $\Pi_C \subset \Pi$ which stabilizes C with $\Pi = R \cdot \Pi_C$ and $C/\Pi_C = M/G$.

The most important polar actions are those for which there are no exceptional orbits and Π_C is trivial. We will refer to them as *Coxeter polar actions*. For such actions, C is in particular a convex set isometric to M/G .

The general data needed for the reconstruction of a Coxeter polar action are as follows: The isotropy groups along C , one for each component of the orbit types. They satisfy *compatibility relations* coming from the slice representations, which themselves are polar. They can be organized in a *graph of groups* denoted by $G(C)$. This group graph generalizes the concept of a group diagram for cohomogeneity one manifolds [GZ1, AA1]. The general *polar data* D needed are $D = (C, G(C))$. The following is our main result:

THEOREM A. *A Coxeter polar action (M, G) is determined by its polar data $D = (C, G(C))$. Specifically, there is a canonical construction of a Coxeter polar G manifold $M(D)$ from polar data $D = (C, G(C))$, and if D is the polar data for a Coxeter polar G manifold M then $M(D)$ is equivariantly diffeomorphic to M .*

For a general polar manifold we construct a G -equivariant cover which is Coxeter polar. Conversely, if M is Coxeter polar with data $(C, G(C))$ and Γ acts isometrically on C preserving the group graph, it extends to a free action on M such that M/Γ is polar. See Theorem 3.4 and Theorem 3.6.

The special class of polar manifolds, where no singular orbits are present is in a sense “diametrically opposite” to that of Coxeter polar manifolds. Those are simply described as

$$M = G/H \times_{\Pi} \Sigma$$

where H is a subgroup of G and Π a discrete subgroup of $N(H)/H$ acting properly discontinuously on Σ . We call these actions *exceptional polar* since this is precisely the class of polar manifolds where the orbits are principal or exceptional, i.e., of the same dimension, thus defining a foliation on M (see Theorem 2.1). These examples also show that *any discrete subgroup of a Lie group can be the polar group of a polar action*, which is our motivation for renaming these groups polar groups, as opposed to generalized Weyl groups, as is common in the literature.

In Theorem A one can include metrics on the section, as was shown in [Me]: If Σ is equipped with a metric invariant under Π , then $M(D)$ has a polar metric with section Σ , extending the metric on Σ . This is in fact crucial in the proof of Theorem A, as well in the construction of a large class of examples.

It is apparent from our general result that Coxeter polar manifolds play a pivotal role in the theory. They are also in various other ways the most interesting ones. For example, one naturally associates to such a manifold a general *chamber system* [Ti2, Ro], which provides a natural link to the theory of buildings. This was used in [FGT] to show that *polar actions on simply connected positively curved manifolds of cohomogeneity at least two are smoothly equivalent to polar actions on rank one symmetric spaces* (here passing from topological to smooth equivalence appeals to our recognition result above). In another direction, the construction of principal \mathbf{L} bundles over a cohomogeneity one manifold with orbit space an interval, crucial for the construction of principal bundles with total space of nonnegative curvature in [GZ1, GZ2], carries over to general Coxeter polar manifolds as a consequence of Theorem A.

THEOREM B. *Let $D = (C, \mathbf{G}(C))$ be data for a Coxeter polar \mathbf{G} manifold M , and let \mathbf{L} be a Lie group. Then for any $\mathbf{G}(C)$ compatible choice of homomorphisms from each $\mathbf{K} \in \mathbf{G}(C)$ into \mathbf{L} , the resulting group graph $\mathbf{L} \times \mathbf{G}(C)$ together with C are Coxeter data, $D' = (C, \mathbf{L} \times \mathbf{G}(C))$ for a Coxeter polar $\mathbf{L} \times \mathbf{G}$ manifold P . Moreover, P is a principal \mathbf{L} bundle over M .*

Examples are polar actions on \mathbb{CP}^n and \mathbb{HP}^n which lift to polar actions on spheres under the Hopf fibration. See also [Mu] for examples of a similar type. This construction is important as well for the special cohomogeneity two case of C_3 geometry in [FGT] where the general theory for buildings brakes down.

Extending the concept of a *hyperpolar* manifold, where the sections are flat (euclidean polar might be a better terminology), we say that a polar manifold M is *spherical polar* resp. *hyperbolic polar* if its sections have constant positive resp. negative sectional curvature.

Using Morse theory we provide a connection between geometric ellipticity and topological ellipticity. Recall that a topological space is called rationally elliptic if all but finitely many homotopy groups are finite. Rationally elliptic manifolds have many very restrictive properties, see [FHT]. The following can be viewed as an extension of the main result for cohomogeneity one manifolds in [GH2]:

THEOREM C. *A simply connected compact spherical polar or hyperpolar manifold M is rationally elliptic.*

We actually prove the stronger property that the Betti numbers of the loop space grow at most polynomially for any field of coefficients (instead of just rational coefficients). It is interesting to compare Theorem C with the (extended) Bott conjecture [GH1, Gr] which states that a manifold with (almost) nonnegative curvature is rationally elliptic. It is thus also a natural question if one can construct metrics with nonnegative curvature on spherical polar or hyperpolar manifolds, possibly under special conditions on the codimensions of the orbits, as was done in the cohomogeneity one case in [GZ1].

Our main recognition theorem above leads to interesting *general surgery constructions*, in particular connected sums, showing how rich the class of polar manifolds indeed is. As an interesting example we will see that $\mathbb{S}^n \times \mathbb{S}^n \# \mathbb{S}^n \times \mathbb{S}^n$ supports a polar $\mathrm{SO}(n) \times \mathrm{SO}(n)$ action of cohomogeneity two, and metric with section $\mathbb{S}^1 \times \mathbb{S}^1 \# \mathbb{S}^1 \times \mathbb{S}^1$ of constant negative curvature. If $n \geq 3$ one can use the proof of Theorem C to give a geometric proof of the well known fact [FHT] that $\mathbb{S}^n \times \mathbb{S}^n \# \mathbb{S}^n \times \mathbb{S}^n$ is rationally hyperbolic.

Another rich class of polar actions is the case of general non-exceptional actions by a torus with 2-dimensional quotient. This gives e.g. rise to polar actions on connected sums of arbitrarily many copies of: $\mathbb{S}^2 \times \mathbb{S}^2$ or $\pm \mathbb{CP}^2$ in dimension 4, $\mathbb{S}^3 \times \mathbb{S}^2$ in dimension 5, and $\mathbb{S}^4 \times \mathbb{S}^2$ or $\mathbb{S}^3 \times \mathbb{S}^3$

in dimension 6. In fact in most cases each such manifold admits infinitely many inequivalent polar actions.

The paper is organized as follows. In Section 1 we recall known properties of polar actions. In Section 2 we develop in detail the geometry of the polar group, its reflection subgroups and their chambers. In Section 3 we prove Theorem A and the reconstruction for general polar manifolds. Finally in Section 4 we discuss various applications and examples and prove Theorem C.

1. PRELIMINARIES

As a natural generalization of *polar representations*, classified by Dadok in [Da], Palais and Terng [PT1] coined a proper isometric action $G \times M \rightarrow M$ to be *polar* if there is an embedded submanifold of M intersecting all G orbits orthogonally. It is common nowadays to work with the following extension since it, e.g., includes all cohomogeneity one manifolds.

Definition 1.1. A proper smooth G action on a (connected) manifold M is called *polar* if there is a G invariant complete Riemannian metric on M and an immersion $\sigma : \Sigma \rightarrow M$ of a connected complete manifold Σ , whose image intersects all G orbits orthogonally.

To be more specific, we require that each G orbit intersects $\sigma(M)$ and $\sigma_*(T_p \Sigma)$ is orthogonal to $T_{\sigma(p)}(G \cdot \sigma(p))$ for all $p \in \Sigma$. Here σ is referred to as a *section* of the action, and M simply as a *polar manifold*. We will implicitly assume that σ has no subcover section. It is an interesting question if the immersion σ has to be injective, at least if M is simply connected.

Note that if G acts polar, then the identity component G_o acts polar as well with the same section, since one easily sees that $\cup_{g \in G_o} \Sigma$ is open and closed in M . But disconnected groups occur naturally, e.g., as slice representations.

We note that the following theorem, due to H. Boualem in the more general situation of singular Riemannian foliations [Bo] (see also [HLO] for the case of group actions), gives an alternate definition, and motivation, to a polar manifold:

THEOREM 1.2. *The horizontal distribution associated with an isometric action is integrable on the regular part if and only if the action is polar.*

There are trivial polar actions where G is discrete and hence $\Sigma = M$, as well as the case where G acts transitively and hence Σ a point. We call such actions *improper* polar actions. We will usually assume that the polar action is proper, although improper ones sometimes occur, e.g., as slice representations of the G action.

In the following we denote by M_o the regular points in M , and by Σ_o those points whose image is in M_o . Recall that M_o is open and dense in M and M_o/G is connected. For any $q \in M$, the isotropy G_q acts via the so called slice representation on the orthogonal complement of the orbit $T_q^\perp := (T_q(G \cdot q))^\perp$. The image $S_q := \exp_q(B_\epsilon)$ in M of a small ball $B_\epsilon \subset T_q^\perp$ is called the slice at q . By the slice theorem $G \times_{G_q} B_\epsilon$ is equivariantly diffeomorphic to a tubular neighborhood of the orbit $G \cdot q$.

Fix a section $\sigma : \Sigma \rightarrow M$, a point $p \in \Sigma_o$, and let $H = G_{\sigma(p)}$ be the isotropy group corresponding to the principal orbit $G \cdot \sigma(p)$. Let $G_{\sigma(\Sigma)} \subset G$ be the stabilizer of the image, and $Z_{\sigma(\Sigma)}$ the centralizer (i.e. fixing $\sigma(\Sigma)$ pointwise). We refer to $\Pi := G_{\sigma(\Sigma)} / Z_{\sigma(\Sigma)}$ as the *polar group* associated to the section σ (the corresponding group was called the “generalized Weyl” group in [PT1]). Note that for improper polar actions $\Pi = G$ when $\Sigma = M$ and $\Pi = \{1\}$ when Σ is a point.

The usual properties of polar actions for which the section is embedded, easily carry over to the immersed case, though care needs to be taken since σ may not be injective. For the convenience of the reader, we indicate the details.

PROPOSITION 1.3. *Let G act on M polar with section $\sigma: \Sigma \rightarrow M$ and polar group Π and projection $\pi: M \rightarrow M/G$. Then the following hold:*

- (a) *The image $\sigma(\Sigma)$ is totally geodesic and determined by the slice of any principal orbit.*
- (b) *The polar group Π is a discrete subgroup of $N(H)/H$ and acts isometrically and properly discontinuously on Σ such that σ is equivariant. Furthermore, $\pi \circ \sigma$ induces an isometry between Σ/Π and M/G and thus M/G is an orbifold.*
- (c) *$\Sigma_o := \sigma^{-1}(M_o)$ is open and dense in Σ and σ is injective on Σ_o . The polar group Π acts freely on Σ_o and $\pi \circ \sigma$ induces an isometric covering from each component of Σ_o onto M_o/G .*
- (d) *The slice representation of an isotropy group G_q , $q \in \sigma(\Sigma)$, is a polar representation, and for each $p \in \sigma^{-1}(q)$ the linear subspace $\sigma_*(T_p\Sigma) \subset T_q^\perp \subset T_qM$ is a section with polar group the (finite) isotropy group Π_p .*

Proof. If $p \in \Sigma_o$, then for dimension reasons $T_{\sigma(p)}^\perp = \sigma_*(T_p\Sigma) \subset T_{\sigma(p)}M$. Thus, if c is a geodesic in Σ with $\gamma(0) = p$, $\sigma(c)$ is horizontal on the regular part and hence a geodesic in M_o since $\pi: M_o \rightarrow M_o/G$ is a Riemannian submersion. The non-regular points on $\sigma(\gamma)$ are isolated. Indeed, the set of regular points is clearly non-empty and open. If $q = \gamma(t_0)$ lies in its closure, the slice theorem at q , since $\sigma(\gamma) \subset S_q$, implies that the isotropy groups at $\sigma(\gamma)(t_0 - \epsilon)$ and $\sigma(\gamma)(t_0 + \epsilon)$ are the same, which proves our claim. This implies that Σ_o is dense in Σ and hence $\sigma(\Sigma)$ is totally geodesic. Since σ has no subcovers, $\sigma_*(T_p\Sigma)$ determines the immersion. Notice this also implies that if $g \in G$ satisfies $g_*(\sigma_*(T_p\Sigma)) = \sigma_*(T_q\Sigma)$ for some $p, q \in \Sigma$, then $g \in G_{\sigma(\Sigma)}$ with $gp = q$. Similarly, $\sigma: \Sigma_o \rightarrow M_o$ is injective and Π acts freely on Σ_o .

By the slice theorem the principal isotropy at $G_{\sigma(p)} = H$ acts trivially on $S_{\sigma(p)} \subset \sigma(\Sigma)$ and since Σ is totally geodesic, H acts trivially on $\sigma(\Sigma)$. In fact, by the slice theorem again, the centralizer $Z_{\sigma(\Sigma)} = Z_{\sigma(\Sigma_o)} = H$. Since H is thus the ineffective kernel of the action of $G_{\sigma(\Sigma)}$ on $\sigma(\Sigma)$, H is normal in $G_{\sigma(\Sigma)}$, i.e., $G_{\sigma(\Sigma)} \subset N(H)$. Next, let $h \in G_{\sigma(\Sigma)}$ be close to the Id. By the slice theorem the orbits in a tubular neighborhood of the regular point $\sigma(p)$ intersect the slice in a unique point and thus $h\sigma(p) = \sigma(p)$, i.e. $h \in H$. Thus H is open in $G_{\sigma(\Sigma)}$ and hence Π is discrete in $N(H)/H$.

The polar group Π acts on $\sigma(\Sigma)$ by definition and preserves $\sigma(\Sigma_o) \subset M_o$. Since $\sigma: \Sigma_o \rightarrow M_o$ is injective, the action lifts to an isometric action on Σ_o such that σ is Π equivariant. Since the action is isometric and Σ_o is dense, $\gamma \in \Pi$ extends to a metric isometry on Σ and hence a smooth isometry. By the same argument as above, $\Pi \subset \text{Iso}(\Sigma)$ is a discrete subgroup. That it acts properly discontinuously is a direct consequence of properness of the G action. Indeed, since the G orbits are closed, so are the Π orbits, hence preventing accumulation points.

We now show that $\pi \circ \sigma$ induces an isometry $\Sigma_o/\Pi \simeq M_o/G$, which by continuity implies $\Sigma/\Pi \simeq M/G$. Let $p_i \in \Sigma_o$ with $\sigma(p_2) = g\sigma(p_1)$. Then $g_*\sigma_*(T_{p_1}\Sigma) = \sigma_*(T_{p_2}\Sigma)$. Hence $g \in G_{\sigma(\Sigma)}$ and thus there exists a unique $\gamma \in \Pi$ with $p_2 = \gamma p_1$. Furthermore, $\pi \circ \sigma: \Sigma_o \rightarrow M_o/G$ is a local isometry, in fact a covering onto M_o/G .

To see that the slice representation of $K := G_{\sigma(p)}$ on $T_{\sigma(p)}^\perp$ is polar with sections $\sigma_*(T_p\Sigma) \subset T_qM$, take a geodesic c in Σ starting at p . Then by the slice theorem the isotropy along the geodesic $\sigma(c(0, \epsilon))$ is constant, and is equal to the isotropy of the slice representation along $t\sigma_*(c'(0))$ for $t \in (0, \epsilon)$. As $t \rightarrow 0$, it follows that the K orbit $K \cdot (t\sigma_*(c'(0)))$ is orthogonal to $\sigma_*(T_p\Sigma)$. In addition, we need to show that each K orbit meets the linear subspace $\sigma_*(T_p\Sigma)$.

Equivalently, $K \cdot s = G \cdot s \cap S_{\sigma(p)}$ with $s \in S_{\sigma(p)}$ meets $\sigma(B_\epsilon(p)) \cap S_{\sigma(p)}$. To see this, pick a regular point $r \in \Sigma_\circ$ near p . Now connect the two orbits $G \cdot \sigma(r)$ and $G \cdot s$ by a minimal geodesic c in M , which we can assume starts at $\sigma(r)$ and is hence tangent to $\sigma_*(T_p \Sigma) = T_{\sigma(p)}(G \cdot \sigma(p))^\perp$. Thus $c \subset \sigma(\Sigma)$, which implies that $G \cdot s$ and hence $K \cdot s$ meets $\sigma(\Sigma)$. Finally, the polar group of $\sigma_*(T_p \Sigma)$ is clearly Π_p . \square

We will call Π_p the *local polar group*. Notice that Π_p is in general smaller than the stabilizer $\Pi_{\sigma(p)}$ (if $\sigma^{-1}(q)$ has more than one point), and that Π acts transitively on $\sigma^{-1}(q)$.

It is interesting to note that for a general group action, the quotient M/G is an orbifold if and only if the action is infinitesimally polar, i.e., all slice representations are polar [LT].

If a polar action by G acts linearly and isometrically on a vector space V , it is called a polar representation. If G/K is a symmetric space with K connected, the isotropy action of K on the tangent space is well known to be polar, the section being a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{k}^\perp \subset \mathfrak{g}$. Such representations are called *s-representations*. Notice that the cohomogeneity of the action of K is the rank of the symmetric space. The action of K of course also induces a polar action on the unit sphere in V , with cohomogeneity one less than the rank.

The remarkable result by Dadok [Da] (see also [EH1] for a conceptual proof if the cohomogeneity is at least 3) is a converse. For this, first recall that two actions with the same orbits are called orbit equivalent.

PROPOSITION 1.4. *Let G be a connected Lie group acting linearly on V .*

- (a) *The G action on V is polar if and only if it is orbit equivalent to an s-representation.*
- (b) *If the G action is polar, the polar group is a finite group generated by linear reflections.*

Notice that orbit equivalent actions have the same Weyl group (as an action on Σ). Hence part (b) follows from (a) since this fact is well known for symmetric spaces. In this case the polar group coincides with the usual Weyl group of a symmetric space. We reserve the word Weyl group for this particular kind of polar group. Thus, if $K = G_{\sigma(p)}$, the slice representation of K_0 is an s-representation and its polar group $\Pi_p \cap K_0$ satisfies all the well known properties of a Weyl group, and in particular is generated by linear reflections with fundamental domain a (Weyl) chamber. The full local polar group Π_p is a finite extension of this Weyl group.

Remark. If two connected groups K and K' are orbit equivalent polar representations, with K being an s-representation, then it was shown in [EH1] that $K' \subset K$ and in [EH2, Ber, FGT] one finds a list of all connected subgroups of s-representations which are orbit equivalent.

For metrics we have the following natural but highly nontrivial result due to R. Mendes [Me]:

THEOREM 1.5 (Metrics). *Let M be a polar G manifold with section $\sigma: \Sigma \rightarrow M$ and polar group Π . Then for any Π invariant metric g_\circ on Σ there exists a G invariant metric g on M with section Σ , such that $\sigma^*(g) = g_\circ$.*

2. POLAR GROUP, REFLECTION SUBGROUPS AND CHAMBERS

To begin a more detailed analysis of the action by the polar group we derive more information about the orbit space $M/G = \Sigma/\Pi$. This in particular will reveal the existence of an important

normal subgroup $R \subset \Pi$ generated by *reflections*, which typically is non-trivial. In particular, the presence of *singular orbits*, i.e., orbits with dimension lower than the regular orbits, will yield the existence of a non-trivial normal reflection subgroup $R_s \subset \Pi$. The associated concept of *mirrors* lead to the presence of open *chambers*, whose length space completion and the isotropy groups along them are the crucial ingredients for the recognition and reconstruction.

We refer to polar actions with no singular points simply as *exceptional polar* actions. For those we have the following characterization:

THEOREM 2.1 (Exceptional Polar). *An action $G \times M \rightarrow M$ without singular orbits is polar if and only if M is equivariantly diffeomorphic to an associated bundle $G/H \times_{\Pi} \Sigma$, where Π is a discrete subgroup of $N(H)/H$ acting isometrically and properly discontinuously on a Riemannian manifold Σ . The image of $(eH) \times \Sigma \subset G/H \times \Sigma$ is a section and Π is the polar group.*

Proof. We start with the Riemannian manifold $M = G/H \times_{\Pi} \Sigma$ on which G acts on the first component from the left. Here (gH, p) is equivalent to $(gn^{-1}H, np)$ for $n \in \Pi \cdot H \subset G$. Since $N(H)/H$ acts freely on G/H on the right, $G/H \times_{\Pi} \Sigma$ is a manifold. The image of (e, Σ) is a section if and only if the metric is induced by a warped product metric of a family of left G right H invariant metrics on G/H with the given metric on Σ . As for the isotropy groups of this action we have $G_{[(eH, p)]} = \Pi_p$, where Π_p is the stabilizer of p . Since Π_p , being a discrete subgroup of $O(T_p\Sigma)$, is finite, all orbits are exceptional.

Conversely, assume that G acts polar on M with section Σ , polar group Π and with all orbits exceptional. The section through every point is unique and hence σ is injective. We claim that $\pi: G \times \Sigma \rightarrow M$, $(g, p) \rightarrow g\sigma(p)$ induces an equivariant diffeomorphism of $G/H \times_{\Pi} \Sigma$ with M . Clearly π is onto. If $p, p' \in \Sigma$ and $g, g' \in G$ with $g\sigma(p) = g'\sigma(p')$, then $(g')^{-1}g\sigma(p) = \sigma(p')$. By uniqueness of the sections, $(g')^{-1}g = \gamma \in \Pi$ and since σ is injective $p' = \gamma p$ and $g' = g\gamma^{-1}$. This proves our claim. \square

Remark. (a) Thus any discrete subgroup of a Lie group is the polar group of some (exceptional) polar manifold.

(b) It follows that a principal G bundle $P \rightarrow B$ with G compact is polar if and only if it has finite structure group. Note that in this case a section is not a section in the principal bundle sense, but rather a “multiple valued” section: It is locally a section, and in fact a cover of the base.

(c) Observe that an exceptional G manifold M (with G compact) has an invariant metric of non-negative curvature if and only if Σ has a Π invariant metric of nonnegative curvature

In the remainder of the paper we will be interested in polar manifolds which are not exceptional.

We will use the following notation: M_o is the *regular* part of M , i.e., the union of all principal orbits, M_e is the *exceptional* part of M , i.e., the union of all non-principal orbits with the same dimension as the principal orbits, and M_s is the *singular* part of M , i.e., the union of all lower dimensional orbits. M_o is open and dense in M and M/G . Recall that $\Sigma_o = \sigma^{-1}(M_o)$ is also open and dense in Σ .

The maximal leaf of the (smooth) horizontal distribution in M_o through a point $p \in M_o$ is the connected component of Σ_o corresponding to the (unique) section Σ with $p \in \sigma(\Sigma)$. From now on, we will denote this component by $\Sigma_o(p)$. Moreover, $\sigma: \Sigma_o(p) \rightarrow M$ is 1-1 and $\pi \circ \sigma: \Sigma_o(p) \rightarrow M_o/G$ is an isometric covering.

We denote by M^K the fixed point set of K (a totally geodesic submanifold) and by $M_{(K)}$ the orbit type, i.e. the union of all orbits with isotropy conjugate to K . The image of any set in M we denote by a $*$, e.g., $M^* = M/G$ and $M_{(K)}^* = M_{(K)}/G$.

Remark. In general, recall that the *core* of a G manifold M is the subset $C(M) \subset M^H$ of the fixed point set of H having non-empty intersection with the regular part M_o (cf. [GS2]), and the *core group* $N(H)/H$ its stabilizer. Any connected component $C_o(M)$ of the core together with the induced action by $G^* := G_{C_o(M)}/H \subset N(H)/H$ has the same significance as the core itself. Since clearly $\sigma(\Sigma) \subset C_o(M)$ for a core component $C_o(M)$, it follows that M is polar if and only if the action of G^* on $C_o(M)$ is polar, and in this case the polar groups of $C_o(M)$ and of M coincide. The principle isotropy of G^* is trivial, but the group G^* itself is in general not connected. Notice also that singular isotropy of G can become exceptional for G^* .

We first recall some facts about the geometry of M/G that follow from the slice theorem. If S is a slice at $q \in M$ and $K = G_q$, then $M_{(K)} \cap S = M^K \cap S$. A neighborhood of the point $p^* = G/K$ in M^* is isometric to S/K and hence $M^K \cap S$ is isometric to a neighborhood of p^* in $M_{(K)}^*$. In particular, each component of $M_{(K)}^*$ is a totally geodesic (but typically not complete) submanifold in the quotient M^* . Furthermore, if $c_{[a,b]}$ is a minimal geodesic between orbits, then $G_{c(t)} = Z_{\text{Im}(c)}$ for all $t \in (a, b)$ since otherwise c could be shortened. Hence M_o^* is convex, and M^* is stratified by (locally) totally geodesic orbit types.

Let $K = G_q$ be an isotropy group where only the principal isotropy group H is smaller. We call such an isotropy group an *almost minimal* isotropy group. Then the action of K on the sphere orthogonal to $T_q(M^K \cap S) \subset T_q S$ has only one orbit type, namely H . As is well known, see, e.g., [Br], p.198, this implies that the action of K on the sphere is either transitive, or it acts freely, and K_o is one of the Hopf actions. But if G is a polar action, the slice representation by K as well as K_o is polar. Since the horizontal space for the Hopf actions is not integrable, it follows that if K acts freely, then it must be finite, i.e., $G \cdot q$ is an exceptional orbit.

If K acts transitively, $q^* \in M^*$ is a boundary point and $M^K \cap S \subset M^*$ is part of the boundary in $M^* = \Sigma^*$. Thus $M_{(K)}^*$ is part of the boundary and a connected component of $M_{(K)}^*$ is called an *open face* and its closure a *face* of the boundary.

We now observe that $M_s^* \subset \partial M^*$. Indeed, if L is a singular isotropy of G , the slice representation of L_o is polar and hence has singular isotropy. By induction, L contains an almost minimal K with K/H not finite and hence $M_{(L)}^*$ lies in the closure of $M_{(K)}^*$. Exceptional orbits can also be part of the boundary: If K is almost minimal with $K/H \simeq \mathbb{Z}_2$ and if the fixed point set of the involution has codimension 1 in the slice, then $M_{(K)}^*$ is a face. An exceptional orbit for which K contains such an involution lies in ∂M^* as well. Of course there can be interior exceptional orbits if M is not simply connected, but their codimension is at least 2. We remark that ∂M^* is also the boundary of M^* in the sense of Alexandrov geometry. Summarizing:

- M^* is convex and stratified by totally geodesic orbit types.
- $M_s^* \subset \partial M^*$ and interior exceptional orbits have codimension at least 2.

As for the geometry of the discrete group action of Π on Σ , (see, e.g., [Dav]), it is common to consider the normal subgroup R consisting of *reflections*. Here r is called a *reflection* if the fixed point set M^r has a component Λ of codimension 1, a totally geodesic reflecting hypersurface called a *mirror* for r . Notice that we do not assume that Λ separates Σ into two parts, as in [Dav]. In particular, M^r can contain several reflecting hypersurfaces (and components of lower dimension). Note though that if Σ is simply connected, degree theory implies that reflections

separate. On the other hand, sections are typically not simply connected, even if M is. See also [AA2] for a general discussion of reflection groups, and reflections groups in which reflections separate.

We denote the set of all mirrors in Σ by \mathfrak{M} . A connected component of the complement of the set of all mirrors in Σ is called an *open chamber*, and denote by c . The closure $C = \bar{c}$ (in Σ) of an open chamber will be referred to as a *closed chamber*, or simply as a chamber of Σ . If $\Lambda \in \mathfrak{M}$, we call the components of the intersection $C \cap \Lambda$ which contain open subsets of Λ a *chamber face* of C . We can provide each chamber face with a label $i \in I$ and will denote the face by F_i and the corresponding reflection by r_i . Note though that different faces can correspond to the same reflection, i.e., possibly $r_i = r_j$.

R (and hence Π) acts transitively on the set of open chambers. In our generality, however, there may be non trivial elements $\gamma \in \Pi$ (even in R) with $\gamma(C) = C$. We denote the subgroup of such elements by Π_C (respectively R_C). Clearly then $\Sigma/\Pi = C/\Pi_C$ and one easily sees that the inclusion $\Pi_C \subset \Pi$ induces an isomorphism $\Pi_C/R_C \simeq \Pi/R$.

Recall that we denote by $\Sigma_o(p)$ the component of Σ_o containing p . We thus have:

- $C = \text{cl}(\Sigma_o(p))$ is locally the union of convex sets with $\partial C = \cup_{i \in I} F_i$. Furthermore, $M^* = C/\Pi_C$, $\partial M^* = \partial C/\Pi_C$, and $\Pi_C/R_C \simeq \Pi/R$.
- Π_C preserve strata types, and in particular permutes chamber faces, possibly preserving some as well.

A component of an isotropy type $\Sigma_{(\Gamma)}$ of Π (or of R) is contained in $\Sigma^{\Gamma'}$ (as an open convex subset) with Γ' conjugate to Γ since a slice at $p \in \Sigma_{(\Gamma)}$ is a neighborhood of p . It is contained in a mirror if Γ , and hence Γ' , contains a reflection $r \in \Gamma'$ since then $M^{\Gamma'} \subset M^r$. Of course different components of $\Sigma_{(\Gamma)}$ can be contained in different mirrors, or even in the same mirror. In general, there can be isotropy in the interior of c , but the codimension of M^Γ is then at least 2. Let $p \in M^\Gamma$ be contained in the Γ isotropy type of Π and $q = \sigma(p)$. Then $K := G_q$ acts polar on the slice with section $\sigma_*(T_p\Sigma) \subset T_q^\perp$ and polar group $\Pi_p \simeq \Gamma$. The fixed point set of K in the slice is $T_q(M^K \cap S) \subset T_q^\perp$, and since every K orbit meets $\sigma_*(T_p\Sigma)$, it follows that $T_q(M^K \cap S) \subset \sigma_*(T_p\Sigma)$, in fact for every $p \in \sigma^{-1}(q)$. Furthermore, K acts polar on the unit sphere \mathbb{S}^ℓ in $T_q(M^K \cap S)^\perp$, without any fixed vectors. One easily sees that this implies that Γ also has no fixed vector in $\mathbb{S}^\ell \cap \sigma_*(T_p\Sigma)$ and hence $T_q(M^K \cap S)$ is also the fixed point set of Γ , in other words $\sigma(U) = M^K \cap S$ for some neighborhood U of p in M^Γ . Thus

- $\pi \circ \sigma: \Sigma \rightarrow M^*$ takes orbit types of Π to orbit types of G and chamber faces of C to faces of ∂M^* . Furthermore, Π_C acts transitively on the chamber faces in the inverse images of a face, and possibly non-trivial on a chamber face as well.
- For all points $p \in \Sigma$ lying in a component of an orbit type, the isotropy $G_{\sigma(p)}$ is constant, and the slice representations are canonically isomorphic as well.

The constancy of the isotropy will be crucial in our reconstruction.

Let F be a face, $\Pi_{C,F} \subset \Pi_C$ the subgroup that stabilizes it, and $\Pi_{C,p}$ the isotropy at a point $p \in F$. Since $C/\Pi_C = M^*$, the orbit types of the action of $\Pi_{C,F}$ on F are taken to the orbit types in M^* . Thus there exists an open and dense set of points in F , the regular points of the action of $\Pi_{C,F}$ on F , along which the G -isotropy is constant equal to K . We call this the *generic* isotropy of F . Again from $C/\Pi_C = M^*$ it follows that:

- If F is a face with generic isotropy K , then at points p in the interior of F the isotropy $G_{\sigma(p)} = K \cdot \Pi_{C,p}$.

Such a generic isotropy is also an almost minimal isotropy, in particular $K/H = \mathbb{S}^\ell$, and at a generic point p the Π isotropy $\Pi_p \simeq \mathbb{Z}_2$, generated by the reflection r in the face F . Of course $r \in K$, in fact r is the unique involution in $(N(H) \cap K)/H$.

It will be important for us to also consider the larger completion C' of c obtained from the intrinsic length metric on c , and the induced surjective map $C' \rightarrow C$, which induces a one-to-one correspondence between orbit types (but not necessarily components of orbit types). The advantage of considering C' as opposed to C is that for $p \in \partial C$ the tangent cone $T_p C$ minus p may not be connected, but these components are “separated” in C' , i.e., $T_p C'$ minus p is now indeed connected. In fact

- Each point of the boundary $\partial C'$ has as a convex neighborhood the cone over the fundamental domain of the reflection subgroup of the local polar group acting on the normal sphere to the orbit.

Due to this phenomena, C may not be convex (see the example below), though it is clearly a union of locally convex sets. This also means that although C/Π_C is an Alexandrov space, C itself may not be.

Π_C acts on the interior of C and induces an isometric action on the completion C' with $C'/\Pi_C = C/\Pi_C$. We refer to it as the lift to C' . The inverse image of chamber faces in C are called chamber faces of C' . We now construct a Riemannian manifold Σ' on which a Coxeter group acts and in which C' will be a convex subset. To do this, first observe that the angle between any two chamber faces F_i and F_j along a component of a codimension 2 intersection (if it exists) determine the order m_{ij} (infinity if $F_i \cap F_j = \emptyset$) of the rotation $r_i r_j$. Although the intersection may have more than one component, we claim that these integers are the same for each component. Indeed, if p is a generic point in a component of $F_i \cap F_j$, then the action of $G_{\sigma(p)}$ on the sphere \mathbb{S}^ℓ normal to the orbit type is cohomogeneity one. The angle is the length of the interval $\mathbb{S}^\ell / G_{\sigma(p)}$, which is equal to the length of the section (a great circle) divided by the Weyl group. Furthermore, the two non-regular isotropy groups are the generic isotropy groups of F_i and F_j . By constancy, the generic isotropy group of the faces are the same on any other component of $F_i \cap F_j$. But this implies that the Weyl group is the same as well since it is generated by the two reflections in the non-regular isotropy. We can thus associate to C a Coxeter group: $M := \{r_i \mid i \in I \text{ with } r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1\}$. Notice that there is a natural homomorphism $M \rightarrow R$ which is onto since R is generated by the reflections r_i . Thus R is the quotient of a Coxeter group. For polar actions, the corresponding Coxeter matrix has special properties: As is well known, the quotient $\mathbb{S}^\ell / G_{\sigma(p)}^0$ is an interval of length π/n with $n = 2, 3, 4, 6$. The same is true for $\mathbb{S}^\ell / G_{\sigma(p)}$ if there are no exceptional orbits (see, e.g., again Lemma 2.4 below), and possibly half as long in general.

As usual, we can use the Coxeter group M to define a smooth Riemannian manifold Σ' on which M acts by taking the quotient of $M \times C'$ by the equivalence relationship $(gr_i, x) \sim (g, x)$ for $x \in F_i$, $g \in M$ which identifies “adjacent” faces. One easily sees that Σ' is smooth since Σ is, and M acts isometrically on Σ' , with fundamental chamber C' , and simply transitively on chambers (in particular $M_{C'}$ is trivial). We thus have a natural map $\Sigma' \rightarrow \Sigma$, taking M translates of C' to R translates of C , which is an isometric covering, equivariant with respect to the quotient homomorphism $M \rightarrow R$. We can think of Σ' as the “universal” section associated to C .

Note that although combinatorially two polar manifolds may be the same, i.e., same M structure, as manifolds they of course depend on the topological structure of the fundamental domain C . Using this for example will result in very different polar manifolds via our reconstruction.

These concepts are illustrated in the following

Example 2.2. Consider the group $R = D_3 = \langle r_o, r_3 \rangle$ acting on S^2 as well as on \mathbb{RP}^2 , where r_o, r_3 are reflections in two great circles making an angle $\pi/3$:

- On S^2 there are three mirrors and six open chambers. Their closure is the orbit space S^2/R , a spherical biangle with angle $\pi/3$. There are two faces each of which are also a chamber face. Their intersection is the intersection of mirrors and coincides with the fixed point set $\text{Fix}(R)$.
- On \mathbb{RP}^2 there are three mirrors and three open chambers. The closure of an open chamber is a spherical biangle with angle $\pi/3$ where the two vertices have been identified! The stabilizer R_C of a chamber has order two, fixes the “mid point” of C and rotates C to itself, mapping one chamber face to the other. The orbit space has one face with one singular point, the fixed point of R and one interior singular point, the fixed point of R_C . (\tilde{C} is a spherical biangle with angle $\pi/3$ and $R_C = \mathbb{Z}_2$ acts by rotation).

In this case, the reflection group obtained by lifting the reflections in \mathbb{RP}^2 to reflections in S^2 does not contain the antipodal map, the deck group. If we extend it by the antipodal map we get the same action on \mathbb{RP}^2 and the orbit spaces are of course the same as well.

Now consider the manifold M with boundary consisting of six circles, obtained from S^2 by removing a small ϵ ball centered at the soul point of each chamber, and let N denote its quotient by the antipodal map, having three boundary circles. Clearly, $R = D_3$ preserves M and induce an action on N .

- $R = D_3$ acts as a reflection group on the doubles $D(M)$ and $D(N)$ of M and N respectively. On $D(M)$ which is an orientable surface of genus five, $R_C = \mathbb{Z}_2$ acts freely on the chamber C . However, on its quotient $D(N)$, $R_C = \mathbb{Z}_2$ acts freely on the open chamber but fixes two points of its closure (on the corresponding \tilde{C} is isometric to a chamber of $D(M)$, and $R_C = \mathbb{Z}_2$ acts freely on it).

Now consider the group $\Pi = \mathbb{Z}_2 \times \mathbb{Z}_3$ acting on Σ the double of a pair of pants, i.e., on a surface of genus two. Here \mathbb{Z}_2 is given by the reflection in the double, i.e., mapping one pair of pants to the other, and \mathbb{Z}_3 rotates each pair of pants. In this example $R = \mathbb{Z}_2$, there are two chambers a pair of pants, and the stabilizer groups of a chamber C are $R_C = \{1\}$ and $\Pi_C = \mathbb{Z}_3$. In this example, Π_C has two fixed points (as it does on the corresponding \tilde{C}).

If we remove a disc at each fixed point of \mathbb{Z}_3 and join the two boundary circles by a cylinder, we get a new manifold, a surface of genus four with the same group acting, but where Π_C acts freely on the chamber.

- Note that the first example above can be realized concretely as sections with polar groups actions: Let $G = \text{SO}(3)$ acting on S^4 with cohomogeneity one and orbit space an interval of length $\pi/3$. Then the suspended action on S^5 is polar with section S^2 and polar group $\Pi = R = A_2$ the dihedral group of order 6. Its quotient action on \mathbb{RP}^5 has section \mathbb{RP}^2 with the same polar group but here $\Pi_C = \mathbb{Z}_2$ as explained above.

One can refine the description of the reflection group R . In fact, Π contains two natural normal reflection groups, R_s and R_e , the *singular* reflection group and the *exceptional* reflection group respectively. Here R_s is generated by the reflections corresponding to faces consisting of singular orbits, and R_e by the reflections corresponding to faces consisting of exceptional orbits. Clearly, $R = R_s \cdot R_e$. The discussion above for chambers can be carried out for either of the subgroups R_s and R_e . This is particularly useful for R_s , where we will use corresponding notation, such as c_s, C_s and C'_s , and M_s for the associated Coxeter group.

We call a polar action *Coxeter polar* if there are no exceptional orbits and Π_C is trivial, in particular $\Pi = R = R_s$ is a reflection group. Note though that Π itself may not be an abstract Coxeter group, though it is a quotient of the Coxeter group M associated to C . Since $C/\Pi_C = M^*$, we have:

PROPOSITION 2.3. *Let G be a Coxeter polar action on M with polar group $\Pi = R_s$. Then Π acts freely and simply transitively on the set of chambers. The chamber C is a convex set isometric to $\Sigma/\Pi \simeq M/G$ with interior $c = \Sigma_o(p)$ isometric to M_o^* . Furthermore, the isotropy group along the interior of a chamber face F_i is constant equal to K_i .*

We will see in fact that a simply connected polar G manifold is Coxeter polar. This also follows from work of Alexandrino and Töben on singular polar foliations (see [AT] (for G compact) and [A]).

Note that by the usual properties of a symmetric space, an s-representations is Coxeter polar. For a general representation we have:

LEMMA 2.4 (Coxeter Isotropy). *Let (V, L) be a polar representation with chambers C, C_s , and principal isotropy group H . Then*

- (a) C_s is the (Weyl) chamber of the Coxeter polar representation (V, L_0) .
- (b) If (V, L) is Coxeter polar, then L is generated by the isotropy groups K_i of the faces of $C (= C_s)$. Furthermore, the representation is determined, up to linear equivalence, by L, H and $\dim V$.
- (c) (V, L) is Coxeter polar if and only if it is orbit equivalent to (V, L_0) , i.e., L is generated by L_0 and H .

Proof. (a) Clearly an orbit Lv is singular if and only if the orbit L_0v is. Thus for both, the group R_s is the same, and hence have the same fundamental domain. But L_0 is orbit equivalent to an s-representation and hence Coxeter polar.

(b) We start by showing that L is generated by the face isotropy groups K_i . Notice that, since there are no exceptional orbits, all isotropy groups on the interior of the face are equal to K_i . The claim that K_i generate L is proved for general polar actions on simply connected positively curved manifolds in [FGT], and we indicate a proof in our context. For this, fix C and $g \in L$, and let L^* be the group generated by K_i . Using Wilking's dual foliation theorem [Wi], applied to the orbit foliation of L on the unit sphere $S^\ell \subset V$, one obtains (from a suitable piecewise horizontal path from a point in C to a point in gC) a sequence, $C = C_0, C_1, \dots, C_n = gC$, where each C_i is a chamber in some section, and consecutive C_i 's have a wall in common. It follows that $C_{i+1} = k_i C_i$ for some k_i in the isotropy group of that wall. From this description one also sees that each $k_i \in L^*$, and hence $gC = k_1 \cdots k_n C$, with $k_1 \cdots k_n \in L^*$. Since the action is polar, i.e., Π_C is trivial, it follows that $g = k_1 \cdots k_n \in L^*$.

Before proving the second part of (b), we prove part (c). If (V, L) is orbit equivalent to (V, L_0) , then it is clearly Coxeter polar. Conversely, assume that (V, L) is Coxeter polar. As we just saw, the face isotropy K_i generate L . Furthermore, K_i/H is connected since it is a sphere of positive dimension due to the fact that K_i is almost minimal and singular. Thus K_i is generated by $K_i \cap L_0$ and H , which implies that L_0 and H generate L . In particular, L and L_0 have the same principal orbits and hence are orbit equivalent.

To finish, we prove the second claim in part (b). For this, observe that from the classification of symmetric spaces (see ,e.g., [Lo]) it follows that two irreducible s-representations by the same (connected) Lie group S are linearly equivalent if they have the same dimension. The same then clearly holds if the s-representation is reducible. If L is connected, recall that L is orbit equivalent

to an s -representation by some Lie group S and that in fact $L \subset S$. Furthermore, S is uniquely determined by L , since otherwise the two s -representations would be orbit equivalent. This determines the representation of L as well. If L is not connected, we just saw that it is generated by L_0 and H and that the representation of L_0 is determined by $\dim V$. As for the action of H on V , it acts trivially on the section, and orthogonal to it as the isotropy representation of a principal orbit L/H . Thus the representation of L is determined as well. \square

Remark 2.5. If (V, L) is polar, L lies in the normalizer of L_0 in $O(V)$. For an irreducible s -representation, one finds a list of the finite group $N_{O(V)}(L_0)/L_0$ in [Lo] Table 10 (in fact $N_{O(V)}(L_0)$ is precisely the stabilizer of the full isometry group of the simply connected symmetric space corresponding to the s -representation). To see which extensions are Coxeter polar, one needs to determine if a new component has a representative which acts trivially on the section of L_0 . For example, for the adjoint representation of a compact Lie group K , $N_{O(V)}(L_0)/L_0 \simeq \text{Aut}(\mathfrak{k})/\text{Int}(\mathfrak{k}) \cup \{-\text{Id}\}$ and one easily sees that the only extension which is possibly Coxeter polar is the one by $-\text{Id}$. For a simple Lie group this extension is in fact Coxeter polar in all cases but $SU(n)$, $SO(4n+2)$ and E_6 . An orbit equivalent subgroup of an s -representation can of course have a larger normalizer and such normalizers can contain finite groups acting freely on the unit sphere in V . This shows that part (b) of Lemma 2.4 does not hold for general polar representations.

Finally, observe the following. The exponential map at $\sigma(p), p \in \partial C$ is $G_{\sigma(p)}$ equivariant and recall that orbit types are totally geodesic. This implies that it takes faces of the slice representation to faces of C containing p in their closure, and that the reflection group of $G_{\sigma(p)}$ consists of reflection in these faces. Furthermore, the generic isotropy groups of the faces are the same as well, and the tangent cone $T_q C$ is a fundamental chamber for the slice representation. This implies in particular:

- If (M, G) is Coxeter polar, then all slice representations are Coxeter polar and hence $G_{\sigma(p)}$ is generated by the face isotropy groups containing p in their closure. Furthermore, the tangent cone $T_q C$ is the (metric) product of the tangent space to the stratum with a cone over a spherical simplex.

Motivated by this, we call a domain C a *Coxeter domain* if it is a manifold with boundary, stratified by totally geodesic submanifolds, such that at all points of a k dimensional stratum, the tangent cone is the product of \mathbb{R}^k with a cone over a spherical simplex. Notice that for any polar action C' and C'_s are Coxeter domains.

3. RECONSTRUCTION FOR POLAR MANIFOLDS

In the previous section we saw how to reconstruct an exceptional polar manifold, i.e., one for which there are no singular orbits, or equivalently $R_s = \{1\}$, from G/H , a section Σ and the associated polar group $\Pi = R_e \cdot \Pi_C \subset N(H)/H$, where H is the principal isotropy group of regular points in the section.

Reconstruction for Coxeter Polar manifolds

We will now discuss the reconstruction problem for polar manifolds and start with the simpler and most important case of Coxeter polar manifolds. As we shall see, in this case one only needs to know the isotropy groups and their slice representations along a chamber C .

Recall that for a Coxeter polar manifold a chamber C is a fundamental domain (in fact a Coxeter domain) for the polar group $\Pi = R(= R_s)$ and $\pi \circ \sigma: C \rightarrow M/G$ is an isometry. The chamber faces $F_i, i \in I$ of C are uniquely identified with faces of M^* and orbit types of Π are identified with orbit types of G . The interior of each face F_i is an orbit space stratum which by abuse of notation we again denote by F_i . The corresponding isotropy group, which is constant, will be denoted by K_i . Similarly, the interior of intersections $F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_\ell}$ (possibly empty) is a stratum denoted by $F_{i_1, i_2, \dots, i_\ell}$ with isotropy group (constant) denoted by $K_{i_1, i_2, \dots, i_\ell}$. Notice that this is true even if the intersection has more than one component since the slice representations are Coxeter polar and by Lemma 2.4 their face isotropy groups $K_{i_1}, \dots, K_{i_\ell}$ generate $K_{i_1, i_2, \dots, i_\ell}$.

We thus can regard the chamber C , or equivalently M^* , as being *marked* with an associated partially ordered *group graph* where the *vertices* are in one to one correspondence with the isotropy groups $K_{i_1, i_2, \dots, i_\ell}$ along the (non empty) strata, i.e.

$$(3.1) \quad \bullet : F_{i_1, i_2, \dots, i_\ell} \longleftrightarrow K_{i_1, i_2, \dots, i_\ell}$$

Moreover, a vertex $K_{i_1, i_2, \dots, i_\ell}$ of “cardinality” ℓ is joined by an *arrow* to each vertex $K_{j_1, j_2, \dots, j_{\ell+1}}$ of cardinality $\ell + 1$ containing $i_1 i_2 \dots i_\ell$ as a subset and to no other vertices. In other words

$$(3.2) \quad \bullet : K_{i_1, \dots, i_j, \dots, i_{\ell+1}} \longrightarrow K_{i_1, i_2, \dots, i_{\ell+1}}, \quad j = 1, \dots, \ell + 1.$$

For isotropy groups this means that the closure of the strata for $F_{i_1, \dots, i_j, \dots, i_{\ell+1}}$ contains the strata for $F_{i_1, i_2, \dots, i_{\ell+1}}$ and there are no strata in between.

We denote this group graph, or more precisely the marking of C , by $G(C)$. In [GZ1] the group graph was called the group diagram for a cohomogeneity one manifold with orbit space an interval and denoted by $H \subset \{K^-, K^+\} \subset G$. Here K^\pm are simply the markings of the endpoints.

Note that for a general group action one can consider a corresponding graph where, however, the isotropy groups corresponding to components of orbit space strata need to be replaced by their conjugacy classes.

There is a natural length metric on the graph $G(C)$, in which each of the individual arrows has length 1. We will say that a vertex, K is at *depth* d if the distance from K to the principal vertex H is d . The depth of $G(C)$ is defined to be the largest depth of all its vertices. Furthermore, the subgraph of the group graph for G consisting of all vertices and arrows eventually ending at K will be called the *history* of K .

In the case of a Coxeter polar action we have the following *compatibility relations* induced by the exponential map (see Section 2).

- For each vertex $F_{i_1, i_2, \dots, i_\ell}$, the slice representation is Coxeter polar with group graph the history of $K_{i_1, i_2, \dots, i_\ell}$. The tangent cone of C along a point in the strata is a chamber of the slice representation whose marking is induced by the marking on C .

Also notice that the pull back of the metric on an ϵ ball $C(q, \epsilon) \subset C$ gives rise to a metric on an ϵ ball in the fundamental chamber of the slice representation. This metric, if extended via the local polar group to a metric on $T_q \Sigma$, is smooth near the origin.

Motivated by these properties we define a *Coxeter polar data* $D = (C, G(C))$ as follows:

- A Coxeter domain C , i.e., a smooth Riemannian manifold with boundary stratified by totally geodesic submanifolds.
- A *partially ordered group graph* $G(C)$ marking the strata of C and satisfying (3.1), (3.2).

- For each vertex $F_{i_1, i_2 \dots i_\ell}$ with isotropy $K_{i_1, i_2 \dots i_\ell}$ there exists a representation of $K_{i_1, i_2 \dots i_\ell}$ on a vector space V which is Coxeter polar and whose group graph is the history of $K_{i_1, i_2 \dots i_\ell}$.
- There exists a smooth Riemannian metric g_V on a small ball $B_\epsilon(0) \subset V$ invariant under the polar group of the representation such that for all $q \in F_{i_1, i_2 \dots i_\ell}$ the tangent cone $T_q C$ is isometric to the restriction of g_V to a chamber of the representation V .

By Lemma 2.4, the representation V is in fact determined by $K_{i_1, i_2 \dots i_\ell}$ and its history up to linear isomorphism.

We will now see how to reconstruct the G manifold M from these data, and how these data by themselves defines a polar manifold. Note that the latter, by definition involves constructing a (complete) Riemannian metric and a section for the action, restricting to the given metric on C .

THEOREM 3.3 (Coxeter Reconstruction). *Any set of smooth Coxeter polar data $D = (C, G(C))$ determines a Coxeter polar G manifold $M(D)$ with orbit space C . Moreover, if M is a Coxeter polar manifold with data D then $M(D)$ is equivariantly diffeomorphic to M .*

Proof. We start with polar data $D = (C, G(C))$ with a compatibly smooth Riemannian metric on C and will construct $M(D)$ as the union of “tubular neighborhoods” of the “orbits” G/K , analogous to the slice theorem. Start with a vertex $K_{i_1 \dots i_\ell}$ and a point $p \in F_{i_1 \dots i_\ell}$. By the compatibility condition, associated to $K = K_{i_1 \dots i_\ell}$ we have a linear Coxeter polar representation V_K with section Σ_K and a smooth metric on $B_\epsilon(0) \subset \Sigma_K$ invariant under Π_K which on the fundamental domain restricts to the metric on the tangent cone $T_q C$. Lemma 2.4 implies that this representation is uniquely determined by the history of K .

We can now use the metric extension Theorem 1.5 to find a smooth K -invariant metric on the ϵ ball $B_\epsilon^*(0) \subset V_K$ which restricts to the metric on $B_\epsilon(0)$, and such that Σ_K is a section. Notice that although the metric on B_ϵ is not complete, the extension Theorem 1.5 is a local statement about smoothness near a singular orbit and thus can be applied in our situation as well (see the proof in [Me]).

We now consider the associated homogeneous vector bundle $G \times_K V_K$ or more precisely the disc bundle $G \times_K B_\epsilon^*$. We endow $G \times B_\epsilon^*$ with a product metric of a biinvariant metric on G and the above K invariant metric on B_ϵ^* . Under the Riemannian submersion $\pi: G \times B_\epsilon^* \rightarrow G \times_K B_\epsilon^*$ this induces a metric on the tube $M(p, \epsilon) := G \times_K B_\epsilon^*$ and G acts isometrically via left multiplication in the first coordinate. Since the orthogonal complement of a G orbit in $M(p, \epsilon)$ is the orthogonal complement of a K orbit in B_ϵ^* , $\{e\} \times B_\epsilon$ is horizontal in the Riemannian submersion π . Thus the action of G on $M(p, \epsilon)$ is polar with local section $\pi(\{e\} \times B_\epsilon)$ and fundamental domain isometric to $C(p, \epsilon) \subset C$.

Altogether, we have constructed for any $K \in G(C)$ and any point p in the K strata a Coxeter polar manifold $M(p, \epsilon)$ with chamber $C(p, \epsilon)$. Moreover, if for some $L \in G(C)$ and $q \in C(p, \epsilon)$ is also in the L strata, the orbit Gq is canonically identified with G/L with q corresponding to $[L] \in G/L$. For two points p_i with non-empty intersection $C(p_1, \epsilon_1) \cap C(p_2, \epsilon_2)$ this provides a canonical equivariant identification of the corresponding orbits in $M(p_i, \epsilon_i)$. Thus we obtain a G manifold $M(p_1, \epsilon_1) \cup M(p_2, \epsilon_2)$ with section $C(p_1, \epsilon_1) \cup C(p_2, \epsilon_2)$. Clearly then, using a partition of unity associated with a cover $\{C(p, \epsilon)\}$ produces a smooth G manifold $M(D)$. Since the orbits by construction are (closed) embedded sub manifolds and the orbit space is complete, so is $M(D)$. Notice that a section Σ , and the polar group Π with its action Σ , is “developed” via repeated reflection. To see that the metric on the section is complete, we can, e.g., apply Theorem 1.2 since the horizontal distribution on $M(D)_\circ$ is integrable by construction.

To see that the action of G on $M(D)$ is Coxeter polar, first observe that it has no exceptional orbits since by assumption, the slice representations have this property. Furthermore, $M(D)/G = C$, and hence C is a fundamental domain for the polar group Π acting on Σ . By construction Π contains all the reflections in the (chamber) faces of C . Along each stratum, these generate by assumption (the local data are Coxeter) a reflection group of the normal sphere to the stratum and hence act freely and transitively on chambers in the normal sphere. If R is larger than the group generated by the reflections in the faces of C , then a chamber for R would be properly contained in C and hence C could not be the orbit space. Similarly Π_C must be trivial since again C is the orbit space C/Π_C .

Conversely, if D is the Coxeter polar data of a given manifold M , the construction above, and the tubular neighborhood theorem for the G action on M , clearly gives a canonical identification of M with $M(D)$ which is a G equivariant diffeomorphism. \square

Remark. (a) Notice that if we start with a section Σ on which Π acts isometrically with $\Pi_C = \{e\}$, and mark the fundamental domain C with some group graph, the new section in $M(D)$ constructed above may not be equal to Σ . However, all sections are covered by the universal section Σ' associated to the Coxeter matrix M defined in terms of the faces of C and their angles between them (see Section 2). In particular, the section is unique if $\Pi = M$. This happens, e.g., for any Coxeter polar manifold with a fixed point since the polar group is clearly the local polar group at the fixed point.

Notice also that the group graph $G(C)$ is already completely determined by the (almost minimal) isotropy groups K_i since they generate all other vertex groups. Also Π is completely determined by the group graph since it is generated by the reflections in the faces F_i of C . As elements in $N(H)/H$, they are the unique involutions in $(N(H) \cap K_i)/H$ (recall that K_i/H is a sphere). In particular, M is compact if and only if C is compact and Π is finite. On the other hand, one can change the metric on C , or the conjugacy classes of the vertex groups in $G(C)$, without changing the equivariant diffeomorphism type of $M(D)$. But such a change can change the topology of Σ and the polar group Π . It is an interesting question if, via such a change, one can make the section embedded, and if the manifold and group is compact, one can make the section compact as well. This is easily seen to be true for a cohomogeneity one action.

(b) We can of course simply enlarge the group $G \subset L$ to obtain new polar data $D'(C, L(C))$ with the same isotropy groups. The new manifold can be regarded as $M \times_G L$. Conversely, the polar group action is called not primitive (and primitive otherwise) if there exists a G -equivariant map $M(C, G(C)) \rightarrow G/L$ for some subgroup $L \subset G$. The fiber over the coset $[L] \in G/L$ is then polar with data $(C, L(C))$, i.e., all G group assignments are contained in L . The section and polar group is the same for both polar actions.

(c) If we denote the inverse of $\pi \circ \sigma|_C$ by $s: M^* \rightarrow C \subset \Sigma$, then $\sigma \circ s$ provides a “splitting” for $\pi: M \rightarrow M^*$. In particular, the orbit type $N(K)/K$ principle bundles $M^K \cap M_{(K)} \rightarrow M_{(K)}^*$ have a canonical section. See also [HH], where they examined group actions with such a splitting in the topological category, and a reconstruction of the topological space with an action of G as well. But in this category there is no compatibility condition.

(d) We point out that under very special assumptions, this Theorem was claimed (without proof) in the cohomogeneity two case in [AA2].

Reconstruction for General Polar manifolds

We will now address the general case, where both R_s and Π_C are non-trivial (recall that R_C could be non-trivial as well). Rather than considering C we will consider C_s or better its Coxeter

completion C'_s together with the (induced) action by $\Pi_{C'_s}$. We point out that replacing C by C_s in our reconstruction is linked to the properties of linear polar actions in Lemma 2.4. Note that C_s is tiled by copies of C under exceptional reflections and $\Pi_C \subset \Pi_{C_s}$, i.e. Π_{C_s} is nontrivial as well. Using the reconstruction for Coxeter polar manifolds we will now construct a cover which is Coxeter polar.

THEOREM 3.4 (Coxeter Covers). *Given a polar manifold $M(C, \mathbf{G}(C))$ with polar group Π and completion C'_s , there exists a Coxeter polar manifold $M(C'_s, \mathbf{G}'(C'_s))$ with polar group \mathbf{R}_s and a free action by $\Pi'_{C'_s}$, commuting with \mathbf{G} , whose quotient is our \mathbf{G} manifold M .*

Proof. Recall that C'_s is indeed a Coxeter domain. We now need to assign a group graph to C'_s satisfying the compatibility conditions in Theorem 3.3. Recall that to each face F_i we associate the generic isotropy group \mathbf{K}_i of the open chamber face. Consider a non-empty intersection of faces $F_{i_1, i_2, \dots, i_\ell} = F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_\ell}$ and pick a generic point p in its interior. The isotropy $G_{\sigma(p)}$ acts on the slice at p and its generic isotropy groups are (via the exponential map) the isotropy groups $\mathbf{K}_{i_1}, \dots, \mathbf{K}_{i_\ell}$. Let $\mathbf{K}_{i_1 \dots i_\ell} \subset G_{\sigma(p)}$ be the subgroup generated by these generic isotropy groups, and restrict the slice representation to $\mathbf{K}_{i_1 \dots i_\ell}$ as well. By Lemma 2.4, this representation of $\mathbf{K}_{i_1 \dots i_\ell}$ is now Coxeter polar. All together, these isotropy groups define the Coxeter data $D' = (C'_s, \mathbf{G}(C'_s))$ for a polar \mathbf{G} manifold $M' = M(D')$ with orbit space C'_s . One easily checks that the compatibility conditions are all satisfied. Note also that by construction, the polar group of M' is the reflection group \mathbf{R}_s for M , a normal subgroup of $\Pi = \mathbf{R}_s \cdot \Pi_{C_s} \subset \mathbf{N}(\mathbf{H})/\mathbf{H}$ with $\mathbf{R}_s \cap \Pi_{C_s} = (\mathbf{R}_s)_{C_s}$.

Now consider the stabilizer group $\Gamma := \Pi_{C'_s} \subset \Pi$ of C'_s and its action on C'_s as well as on orbits or strata of orbits: Let x, y lie in two strata of C'_s (possibly the same) with generic isotropy $\mathbf{K}_{i_1 \dots i_\ell}$ and $\mathbf{K}_{j_1 \dots j_\ell}$. If $y = \gamma x$ with $\gamma \in \Gamma$, then we of course have $\mathbf{G}_y = \gamma \mathbf{G}_x \gamma^{-1}$, i.e. $\gamma \mathbf{K}_{i_1 \dots i_\ell} \gamma^{-1} = \mathbf{K}_{j_1 \dots j_\ell}$. We therefore have a natural identification of the orbit through x with the orbit through y via $g \mathbf{K}_{i_1 \dots i_\ell} \rightarrow g \gamma^{-1} \mathbf{K}_{j_1 \dots j_\ell}$. This defines an action of Γ on M' commuting with \mathbf{G} . We can choose a metric on M' as in the proof of Theorem 3.3 such that Γ acts by isometries and the metric on C_s is unchanged. Moreover this action is free. Indeed, if $x = \gamma x$ then $\gamma \in \mathbf{N}(\mathbf{K}_{i_1 \dots i_\ell})/\mathbf{K}_{i_1 \dots i_\ell}$. But recall that in general for a homogeneous space \mathbf{G}/\mathbf{L} the action of $\mathbf{N}(\mathbf{L})/\mathbf{L}$ on \mathbf{G}/\mathbf{L} on the right is free.

We now have a free action of Γ on M' and an induced action by \mathbf{G} on the quotient $N = M'/\Gamma$. We claim that (N, \mathbf{G}) is equivariantly diffeomorphic to the action on the given (M, \mathbf{G}) . They both have the same fundamental domain C_s , and completion C'_s , by construction. Also recall that the isotropy in M at $p \in F_{i_1, i_2, \dots, i_\ell}$ is the generic isotropy $\mathbf{K}_{i_1, i_2, \dots, i_\ell}$ extended by the isotropy of Γ at p . But this is also the isotropy of the \mathbf{G} action on N and we can hence define the diffeomorphism orbit wise. \square

Remark 3.5. (a) This also reproves in a concrete way the results of Alexandrino and Töben that polar actions on simply connected manifolds by a connected group have no exceptional orbits, and that such an action is Coxeter polar. Indeed, the existence of exceptional orbits means that $\Gamma = \Pi_{C'_s}$ is non trivial, and hence one gets a cover (which is connected since \mathbf{G} is connected).

(b) One can interpret this even when \mathbf{R}_s is trivial, in which case the ‘‘Coxeter domain’’ $C'_s = C_s = \Sigma$, thus covering exceptional polar actions.

The proof of Theorem 3.4 also shows how to take quotients of Coxeter polar manifolds.

THEOREM 3.6 (Coxeter Quotients). *Let $(M, \mathbf{G}) = M(C, \mathbf{G}(C))$ be a Coxeter polar manifold with polar group Π . Let $\Gamma \subset \mathbf{N}(\mathbf{H})/\mathbf{H}$ be a subgroup that normalizes Π and acts on $(C, \mathbf{G}(C))$, i.e., Γ acts on C isometrically preserving strata and conjugates the corresponding group assignment. Then Γ acts isometrically and freely on M commuting with the action of \mathbf{G} and the quotient $(M/\Gamma, \mathbf{G})$ is polar with polar group $\Pi \cdot \Gamma$.*

Remark 3.7. This also encodes that if a subgroup of Γ preserves a stratum, it must lie in the normalizer of the assigned group. The section in M/Γ is of course simply the image under the cover and hence $\Gamma \cap \Pi$ is the stabilizer group of a chamber in M/Γ and the section of M/Γ has $|\Pi/\Gamma \cap \Pi|$ chambers. We also allow the possibility that $\Gamma \subset \Pi$ is a proper subgroup.

4. CONSTRUCTIONS, APPLICATIONS AND EXAMPLES

Although polar actions may seem rather special and rigid, the construction from data provided in the previous section is amenable to various operations leading to a somewhat surprising flexibility and a wealth of examples. In addition to the trivial process of taking *products*, these operations include general *surgery type constructions*, as well as the construction of *principal bundles*. When combined with the flexibility in the choice of metrics this can be used to derive topological conclusion by means of geometric arguments.

Cutting and Pasting Operations

Consider a polar \mathbf{G} manifold M with an invariant hyper surface E corresponding to a Π invariant separating hyper surface $\Delta \subset \Sigma$ missing 0 dimensional Π strata. Accordingly we have decompositions $M = M_1 \cup M_2$, $\Sigma = \Sigma_1 \cup \Sigma_2$ and $M^* = M_1^* \cup M_2^*$, where M_i are polar \mathbf{G} manifolds with common boundary E , and Σ_i are Π manifolds with common boundary Δ . Here, typically Π does not act effectively on Δ and Π/\ker is the polar group for the section Δ in E . Note also that Δ^* determines Δ via invariance and is automatically perpendicular to the strata it meets. We will also refer to such a Δ^* as a separating hyper surface in $M^* = \Sigma^*$.

Now suppose M' is another polar \mathbf{G} manifold which is separated in the same fashion into two polar \mathbf{G} manifolds M'_1 and M'_2 with common boundary E' . If moreover there is a \mathbf{G} equivariant isometry A from a neighborhood of $E \subset M$ to a neighborhood of $E' \subset M'$, then clearly $M_1 \cup_A M'_2$ is a polar \mathbf{G} manifold as well, with section $\Sigma_1 \cup_A \Sigma'_2$.

Notice that as long as a separating hypersurface Δ^* in M^* is orthogonal to all the strata it meets, the inverse image in M is a smooth separating hypersurface E . The condition that a neighborhood of E is \mathbf{G} isometric to a neighborhood of E' can be achieved via our reconstruction process (including metrics) as long as there is a Π/\ker invariant isometry between neighborhoods of Δ and Δ' . To achieve this modulo a change of metrics on the sections Σ and Σ' it suffices to have a Π/\ker invariant isometry between Δ and Δ' : Indeed, making the geodesic reflections in Δ and Δ' isometries in tubular neighborhoods makes Δ and Δ' totally geodesic. Next one can change the metrics to be product near Δ and Δ' , in both steps keeping the actions by the polar groups isometric (also with respect to a partition of unity).

This in particular proves the following

THEOREM 4.1. *Let M and M' be Coxeter polar \mathbf{G} manifolds with orbit spaces C and C' . Let Δ , respectively Δ' be codimension 1 separating hypersurfaces in C and C' orthogonal to the strata they meet. Suppose there is an isometry $A : \Delta \rightarrow \Delta'$ such that corresponding isotropy data are the same. Then A induces a \mathbf{G} -equivariant diffeomorphism between the codimension*

one sub manifolds of M and M' corresponding to Δ and Δ' . In particular, $M_1 \cup M_2'$ has the structure of a polar G manifold.

Remark. (a) The separating hyper surfaces need not meet any lower strata. This way for example one can glue together two copies of the complement of a tubular neighborhood of a principal orbit (see also [AT]). In fact, this is possible for any orbit as long as the slice representations are equivalent. Another special case is the connected sum along fixed points if the isotropy representations at the fixed points are equivalent.

(b) G equivariant standard surgery can be performed in the following case. Suppose $S^k \subset M^n$ is G invariant sphere in M with trivial normal bundle. Assume $G: \mathbb{R}^{k+1} \oplus \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{k+1} \oplus \mathbb{R}^{n-k}$ is a reducible polar representation, such that when restricted to a tubular neighborhood of $S^k \subset S^n$ is equivalent to the restriction of the G action on M to a tubular neighborhood of S^k . Then the manifold N obtained from M by replacing $S^k \times \mathbb{D}^{n-k}$ with $\mathbb{D}^{k+1} \times S^{n-k-1}$ is a polar G manifold.

Here are some interesting concrete examples applying this type of operation:

Example 4.2. (a) Consider the standard cohomogeneity one action by $SO(n)$ on S^n with two fixed points, section a circle and polar group \mathbb{Z}_2 . The cohomogeneity two product action by $G = SO(n) \times SO(m)$ on $S^n \times S^m$ is polar with section a torus and polar group $\mathbb{Z}_2 \times \mathbb{Z}_2$. By performing a connected sum at fixed points of G it follows from the above that $M = S^n \times S^m \# S^n \times S^m$ admits a polar G action with section $S^1 \times S^1 \# S^1 \times S^1$ and orbit space a right angled hexagon. In particular M admits a polar metric with section of constant negative curvature. We can of course also take further connected sums and obtain a section of any given higher genus.

(b) Consider the standard polar action by T^n on \mathbb{CP}^n (and the one where the action is reversed via the inverse) with $n+1$ fixed points and section \mathbb{RP}^n . Thus $\mathbb{CP}^n \# \pm \mathbb{CP}^n$ admits a polar action and in fact admits a non negatively curved invariant metric (using the Cheeger construction [Ch]) with section $\mathbb{RP}^n \# \mathbb{RP}^n$. Taking further connected sums $M = \mathbb{CP}^n \# \dots \# \mathbb{CP}^n$ yields a polar T^n manifold with section $\mathbb{RP}^n \# \dots \# \mathbb{RP}^n$. If $n = 2$, we can equip M with a metric such that the section has constant curvature 0 or -1 .

(c) In the first example (and similarly in the second) we can do a surgery along a principal orbit on two copies of the action. The section will be the double of $S^1 \times S^1$ minus four discs, one at the center of all four chamber squares of $S^1 \times S^1$. The polar group remains $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) Recall that a symmetric space $M = G/H$ has a natural polar action by H with a fixed point. Thus $M \# M$ has a polar action by H as well.

In general using such constructions with the basic sources coming from cohomogeneity one manifolds and *model examples* coming from polar actions on symmetric spaces (as above) leads to a wealth of interesting examples.

Here are some further (model) examples illustrating our data:

Example 4.3. The case where the section is 2-dimensional is particularly simple, but already gives rise to many interesting examples. This case is also easier geometrically since by the uniformization theorem, and the metric extension theorem, we can assume that Σ has constant curvature and that Π acts by isometries. Thus for a Coxeter polar action we can simply start with a smooth metric on a domain C of constant curvature such that the boundary ∂C is a geodesic polygon. Recall also that the angle between two geodesics in ∂C is one of π/ℓ , with $\ell = 2, 3, 4, 6$ and the corresponding local Weyl group is the dihedral group D_ℓ . The sign of the curvature is determined by applying the Gauss Bonnet theorem to C , in particular, the section is typically hyperbolic.

If K_i are the isotropy groups of the sides, L_i the isotropy groups at the vertices, and H the principle isotropy group, then the compatibility condition simply says that K_i/H are spheres, and that L_i defines a cohomogeneity one action on another sphere, with group diagram given by L_i , the two adjacent sides, and the principle isotropy group H . Notice also that the smoothness condition (i.e. extension to a smooth metric in a neighborhood of $p \in \partial C$ invariant under the local polar group) is now automatic since the metric has constant curvature. Of course the angle at the vertex is also determined by the vertex groups L_i . The marking by the groups also determines the polar group $\Pi \subset N(H)/H$ since it is generated by the reflections in the sides. Thus if C is compact and Π finite, the isotropy groups and the order $|\Pi|$ also determine the genus of the (compact) section, again by the Gauss Bonnet theorem.

We can of course also change the topology of Σ on the interior, and hence the topology of M , via connected sums with any surface and keep the same isotropy groups. But it also becomes clear that for example for a domain C with 3 vertices, the compatibility conditions makes it quite delicate for the group diagram to be consistent since the sides must be the common isotropy groups of three cohomogeneity one actions on spheres (but only after it has been made effective). See e.g. [GWZ] for a list of cohomogeneity one actions on spheres, together with their Weyl group. Notice that in the case when the slice representation is reducible, i.e. the angle between the sides is $\pi/2$ and $\Pi = D_2$, the description is contained in what is called a generalized sum representation in [GWZ].

A typical example of a cohomogeneity two action on a rank one symmetric space is given in Figure 1. The section is an \mathbb{RP}^2 and the polar group is $\Pi = R = C_3/\mathbb{Z}_2$ generated by the 3 reflections in the sides with the further relation coming from the fact that the C_3 action on \mathbb{S}^2 contains the antipodal map. The Coxeter group associated to C is $M = C_3$ and the universal section $\Sigma' = \mathbb{S}^2$.

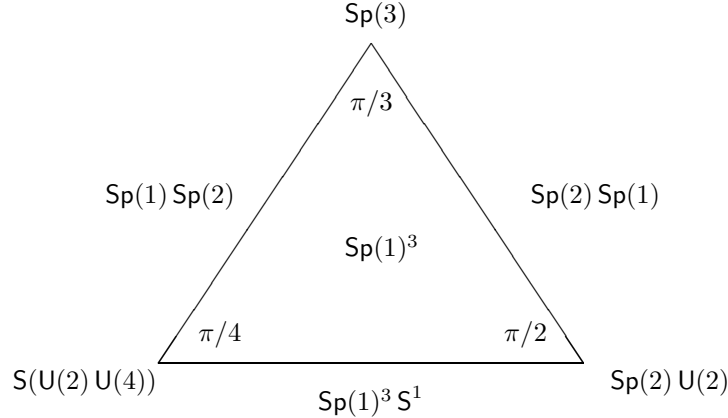


Figure 1. Cohomogeneity two polar action of $SU(6)$ on \mathbb{CP}^{14} .

An example of an action on a symmetric space of rank > 1 , where the sections have to be flat, is given in Figure 2. The vertices are fixed points and the slice representations are the well known irreducible polar action of $SO(3)$ on \mathbb{R}^5 . $O(2)$, $O'(2)$, $O''(2)$ are the 3 different block embeddings of $O(2)$ in $SO(3)$. The polar group is the polar group at the fixed point and hence

$\Pi = D_3$ (which in this case is all of $\mathbf{N}(\mathbf{H})/\mathbf{H}$). Notice that here there are only 2 orbits types, besides the principal one, each of which has 3 components, and that the isotropy along the 3 sides must all be different in order for the group diagram to be consistent. We could add an action of $\Pi_C = \mathbb{Z}_3 \subset \Pi$ here that rotates the vertices, which defines a polar action on a subcover of $\mathrm{SU}(3)/\mathrm{SO}(3)$. Notice though that $\mathbb{Z}_2 \subset \Pi$ is not allowed since it is not normal in Π .

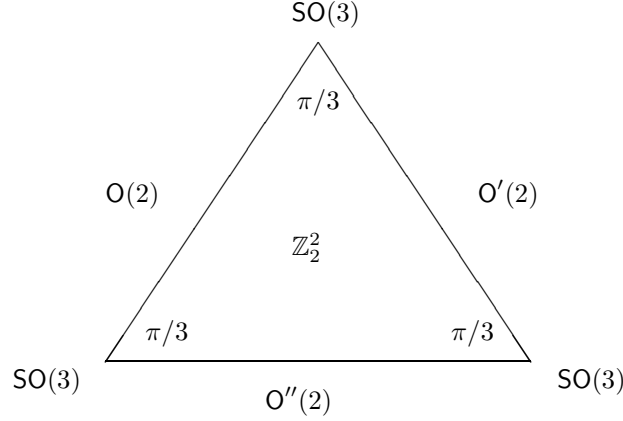


Figure 2. Cohomogeneity two polar action of $\mathrm{SO}(3)$ on $\mathrm{SU}(3)/\mathrm{SO}(3)$.

In the third example, where the polar manifold is not a symmetric space anymore, we replace in the previous example two of the vertex groups by $\mathrm{SO}(4)$ and let $\mathbf{G} = \mathrm{SO}(4)$. The slice representation of $\mathrm{SO}(4)$ at the 2 fixed points is given by the 8 dimensional irreducible polar representation coming from the symmetric space $\mathbf{G}_2/\mathrm{SO}(4)$. Care needs to be taken for the 3 embeddings of $\mathrm{O}(2)$ in $\mathrm{SO}(4)$. To describe this, it is easier to let $\mathbf{G} = \mathbf{S}^3 \times \mathbf{S}^3$ act ineffectively. Then for the cohomogeneity one action by $\mathbf{S}^3 \times \mathbf{S}^3$ on \mathbb{S}^7 the isotropy groups are $\mathbf{H} = \Delta \mathbf{Q}$ and $\mathbf{K}_- = e^{(i\theta, 3i\theta)} \cdot \mathbf{H}$, $\mathbf{K}_+ = e^{(j\theta, j\theta)} \cdot \mathbf{H}$ up to conjugacy by an element in $\mathbf{S}^3 \times \mathbf{S}^3$ (see e.g. [GWZ]). For the cohomogeneity one action by \mathbf{S}^3 (effectively $\mathrm{SO}(3)$) on \mathbb{S}^4 they are $\mathbf{K}_- = e^{i\theta} \cdot \mathbf{H}$, $\mathbf{K}_+ = e^{j\theta} \cdot \mathbf{H}$, $\mathbf{H} = \mathbf{Q}$ up to conjugacy, and the embedding of the third vertex group $\mathrm{SO}(3) \subset \mathrm{SO}(4)$ corresponds to the embedding $\Delta \mathbf{S}^3 \subset \mathbf{S}^3 \times \mathbf{S}^3$. Thus a consistent choice for the isotropy of the sides is e.g. $e^{(i\theta, 3i\theta)} \cdot \mathbf{H}$, $e^{(j\theta, j\theta)} \cdot \mathbf{H}$ and $e^{(k\theta, k\theta)} \cdot \mathbf{H}$. The polar group Π is now D_6 , and since $\Sigma' = \Sigma$ one easily sees that the genus of the section is 2. Notice that the section here must be orientable since C is and Π acts simply transitively on chambers.

Notice though that in Figure 3 it is not possible to have one or three fixed points since there is no compatible choice of side groups. On the other hand, we can easily generalize this example to a domain C with arbitrarily many sides, or by adding handles in the interior. In order for the group diagram to be consistent, the fixed points by $\mathrm{SO}(4)$ must come in adjacent pairs, and the remaining vertices have isotropy $\mathrm{SO}(3)$. This gives rise to a large class of 8 dimensional polar manifolds.

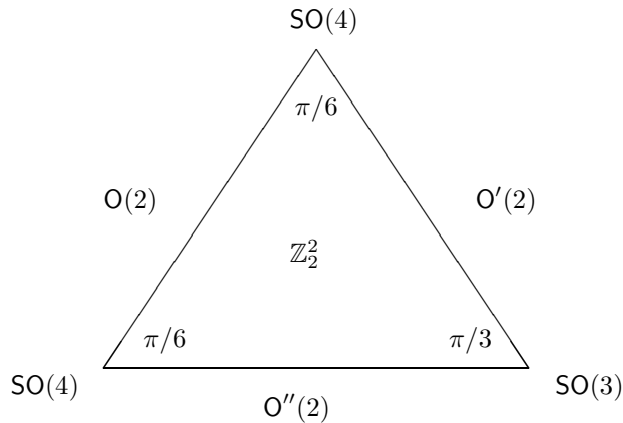


Figure 3. Cohomogeneity two polar action of $SO(4)$ on M^8 .

Example 4.4. We now claim that every T^n action on a compact simply connected $(n+2)$ -dimensional manifold M is polar, if it has singular orbits. Indeed, by the theory of Orlik-Raymond [OR], every such torus action is classified by the following data: The quotient M/T^n is a 2-disk with k edges and k vertices. The principal isotropy is trivial, the vertices have isotropy T^2 , and the isotropy groups along the sides are S^1 with some slopes, the only condition being that T^2 is the product of the two circles assigned to any two adjacent sides. Two assignments give the same action, if and only if they can be carried into each other by an automorphism of T^n . The slice representations at the vertices are effective representations of T^2 on \mathbb{R}^{n+2} and hence the angles at the vertices are $\pi/2$. But every such representation is polar. Hence all compatibility conditions are satisfied, and we can apply Theorem 3.3 to obtain a polar action on M^{n+2} .

In [OR] it was shown that if $n = 4$, the manifolds are diffeomorphic to connected sums of k copies of $S^2 \times S^2$ or $\pm\mathbb{CP}^2$, but on each such manifold there are infinitely many distinct T^2 actions, the only exceptions being \mathbb{CP}^2 and $\mathbb{CP}^2 \# \mathbb{CP}^2$ where the action is unique.

If $n = 5$ such 5 manifolds are diffeomorphic to S^5 or connected sums of arbitrarily many copies of $S^3 \times S^2$, possibly with the non-trivial S^3 bundle over S^2 as well (see [Oh1]). Again, each space except S^5 , admits infinitely many T^3 actions.

In dimension 6, it was shown in [Oh2] that such a manifold is diffeomorphic to connected sums of arbitrarily many copies of $S^4 \times S^2$, $S^3 \times S^3$, and possibly with the non-trivial S^4 bundle over S^2 as well, again with infinitely many actions in most cases.

Bundle Constructions

Note that the following *quotient operation* holds within the class of polar actions: If (M, G) is polar with section σ , and L a normal subgroup of G acting freely on M , then $(M/L, G/L)$ is polar with section σ/L .

In the language of principal bundles, the polar action of G/L on M/L admits a commuting lift to the total space M . We are interested in such *commuting lifts*. See [GZ2] for a discussion of commuting lifts in the cohomogeneity one situation. Recall also that one obtains a larger

collection of commuting lifts if one allows the action of G on M to be almost effective, i.e G and the principle isotropy group H have a finite central subgroup in common.

Such bundles can be constructed via data as follows.

THEOREM 4.5. *Let $D = (C, G(C))$ be the (almost effective) data for a Coxeter polar G manifold $M(D)$. Let L be a Lie group and choose for each $K \in G(C)$ homomorphisms $\phi_K: K \rightarrow L$ such that $\phi_K|_U = \phi_U$ for any $U \in G(C)$ connected in the graph $G(C)$ to K via the inclusion $U \subset K$. Then for each $K \in G(C)$ we obtain embeddings of K into $G^* = L \times G$ via $k \rightarrow (\phi_K(k), k)$. This defines a group graph $G^*(C)$, and hence the data $D^* = (C, G^*(C))$ for a Coxeter polar G^* manifold $M(D^*)$. Moreover, $M(D^*)$ is a principal L bundle over $M(D)$ and the polar metrics on $M(D^*)$ and $M(D)$ can be chosen such that the projection is a G equivariant Riemannian submersion.*

Proof. Notice that in the group graph $G^*(C)$ the vertex groups K , as well as their slice representations V_K , are unchanged. Thus the necessary compatibility conditions are satisfied and hence we obtain a polar manifold $M(D^*)$. The subgroup $L \simeq L \times \{e\} \subset L \times G$ acts along the orbits $(L \times G)/K$ from the left and the action is free (and isometric) since the embedding of K in $L \times G$ is injective in the second coordinate by construction (since L is normal, the isotropy groups are simply the intersection of $L \times \{e\}$ with K). Thus we have a principle bundle $\pi: M(D^*) \rightarrow M(D^*)/L = B$ and on the base B we obtain an induced metric and isometric action by G , such that π is a G -equivariant Riemannian submersion. The section in $M(D^*)$ is horizontal w.r.t. π and hence it covers an immersed submanifold in B orthogonal to the G orbits. Thus (B, G) is polar, and since $M(D^*)/(L \times G) = B/G$ the fundamental domain is isometric to the given metric on C . One easily checks (orbitwise) that the isotropy groups of the G action on B are the same as that in $M(D)$, i.e. the marking of C for G action on B is the same as that for $M(D)$ and hence Theorem 3.3 implies that B is equivariantly diffeomorphic to $M(D)$. \square

Remark. (a) Unlike in the cohomogeneity one case, if $P \rightarrow B$ is an L principle bundle and G acts polar on B and admits a lift to P that commutes with L , the action of $L \times G$ on P may not be polar.

(c) Such a principle bundle construction was carried out in the topological category in [HH].

Example 4.6. In [GM] Garcia and Kerin showed that every T^2 action on $S^2 \times S^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$, and every T^3 action on $S^3 \times S^2$ or the non-trivial S^3 bundle over S^2 , admits an invariant metric with non-negative sectional curvature. The proof shows that the manifolds can be described by taking a quotient of $S^3 \times S^3$ by a sub torus $S \subset T^4$ and the action is the induced action by T^4/S . Since the action by T^4 is polar in the product metric, it follows that the T^2 and T^3 actions on these 4 and 5 manifolds admit polar metrics with non-negative sectional curvature. Note that this is an example of our “quotient” principle bundle construction. In Figure 4 we depict the group diagram on the 4 dimensional manifolds (see [GM]).

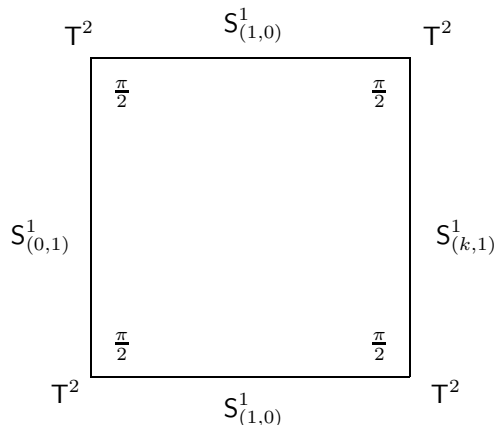


Figure 4. \mathbb{T}^2 polar action on $\mathbb{S}^2 \times \mathbb{S}^2$ (k even) and $\mathbb{CP}^2 \# -\mathbb{CP}^2$ (k odd).

Geometric and Topological Ellipticity and hyperbolicity

Polar manifolds with a flat section are referred to as *hyperpolar* in the literature. More generally we will examine polar manifolds with constant curvature sections, which we will refer to as *polar space forms* or more precisely a *spherical* respectively *hyperbolic* polar manifold if the section has positive respectively negative constant curvature.

Recall that M is called (topologically) elliptic if the Betti numbers of the loop space grow at most polynomially for any field of coefficients. It is well known that simply connected homogeneous spaces and biquotients have this property. If this holds for rational coefficients, the space is called rationally elliptic, which is also equivalent to the condition that all but finitely many homotopy groups are finite. If the manifold is not rationally elliptic, it is called rationally hyperbolic. Rationally elliptic manifolds are severely restricted, see [FHT], and [GH1] in the context of non-negative curvature.

THEOREM 4.7. *Let M be a simply connected closed polar manifold which is hyper polar or spherical polar. Then M is topologically elliptic, in particular rationally elliptic.*

Proof. By Theorem 3.4 (M, G) is Coxeter polar, with a section Σ of curvature 0 or 1. If we fix a Weyl chamber $C \subset \Sigma$, the section Σ is then “tiled” by Π translates of C .

Let $p, q \in C \subset \Sigma$ be regular points. The geodesics starting at p and ending perpendicularly at the orbit $Gq = G/H$ are exactly the critical points for the energy function on the path space F of paths starting at p and ending at Gq . Since the geodesic is perpendicular to the regular orbit Gq , it is perpendicular to all regular orbits, in particular to Gp as well. Hence the geodesic lies in the given section Σ , starting at p and ending at the Π orbit of q . As long as p is not a focal point for Gq , they are all non-degenerate and their indexes are computed by the Morse index theorem. For a generic choice of p and q moreover, these geodesics only cross the walls of the tiles at interior points of the wall. Such points (together with conjugates points at multiples of π when the section has curvature 1) are also precisely the focal points of Gq along geodesic. At such a point the isotropy acts transitively on the normal sphere to the orbit type and hence

the index increases by one less than the codimension of the orbit type. (When projected to the orbit space, these geodesics can be thought of as geodesic “billiards”).

Since there are only finitely many walls in C , and since the universal cover of Σ is \mathbb{R}^n or \mathbb{S}^n with its canonical metric on which the lift of the polar group acts isometrically, it follows that the growth of the Betti numbers (for any field of coefficients) of the path space F is at most polynomial.

This path space in turn is the homotopy fiber of the inclusion map $Gq \rightarrow M$. This suffices to complete the proof if the principal orbit $Gq = G/H$ is simply connected. To see this, consider the fibration $F \rightarrow G/H \rightarrow M$ and the induced sequence of iterated loop spaces

$$\Omega F \rightarrow \Omega(G/H) \rightarrow \Omega M \rightarrow F \rightarrow G/H \rightarrow M$$

where also $\Omega(G/H) \rightarrow \Omega M \rightarrow F$ is a fibration (cf. e.g. [Ha]). Since $\pi_1(G/H) = 0$, we can write $G/H = G'/H'$ with G' compact and simply connected and H' connected. Using the energy function for the biinvariant metric on G' , we see that $\Omega G'$ (which is now connected) is topologically elliptic and from the spectral sequence of the fibration $\Omega G' \rightarrow \Omega(G'/H') \rightarrow H'$ it follows that the same is true for $\Omega(G/H)$. The fibration $\Omega(G/H) \rightarrow \Omega M \rightarrow F$ now shows that ΩM is topologically elliptic as well.

Basically the same reasoning can be applied if G/H has finite fundamental group, say of order ℓ : In this case one replaces G/H by its universal ℓ fold cover G'/H' and the inclusion map $G/H \rightarrow M$ by the composed map $G'/H' \rightarrow G/H \rightarrow M$. For each critical point in F one gets ℓ corresponding critical points in the homotopy fiber F' of $G'/H' \rightarrow G/H \rightarrow M$ each with the same index. In particular our argument above also shows that the Betti numbers of F' grow at most polynomially, and we are done.

If G/H has infinite fundamental group we replace G by $SU(n)$ for n large enough so that $G \subset SU(n)$, and M by $M' := SU(n) \times_G M$ the total space of the bundle with fiber M associated with the principal G bundle $SU(n) \rightarrow SU(n)/G$. Since $SU(n)$ and M are simply connected so is M' . From the fibration $G \rightarrow SU(n) \times M \rightarrow M'$ it is clear that M is topologically elliptic if M' is. Clearly $SU(n)$ acts isometrically on M' when M' is equipped with the quotient metric induced from the product metric on $SU(n) \times M$ where the metric on $SU(n)$ is biinvariant. The orbit space $M'/SU(n)$ is isometric to M/G , and the principal $SU(n)$ orbits are $SU(n)/H$, hence have finite fundamental group. Moreover the $SU(n)$ action on M' is polar with section Σ in one of the fibres of $M \rightarrow M' \rightarrow SU(n)/G$. The proof provided above then shows that indeed M' is topologically elliptic. \square

Remark. In dimension at most 5, a simply connected compact manifold is topologically elliptic if and only if it is diffeomorphic to one of the known manifolds with non-negative curvature, i.e. $\mathbb{S}^n, n \leq 5$, $\pm \mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$, $\mathbb{S}^3 \times \mathbb{S}^2$, $SU(3)/SO(3)$ or the non-trivial \mathbb{S}^3 bundle over \mathbb{S}^2 . (see [PP] for $n = 5$). Thus in these dimensions they are the only manifolds that can admit a polar action with two dimensional section of non-negative curvature (since in this case we can assume it also admits a polar metric with section of constant curvature 0 or 1). It turns out that these are also the only manifolds that admit cohomogeneity one actions [Ho].

In view of the above theorem it is natural to

CONJECTURE. *A simply connected hyperbolic polar manifold is rationally hyperbolic.*

This can be verified for many such manifolds using the ideas in the proof of Theorem 4.7. For example, recall that $M = \mathbb{S}^n \times \mathbb{S}^n \# \mathbb{S}^n \times \mathbb{S}^n$ supports a polar action by $SO(n) \times SO(n)$ with section $\mathbb{S}^1 \times \mathbb{S}^1 \# \mathbb{S}^1 \times \mathbb{S}^1$ of constant negative curvature. The codimension of any stratum (neither

principal nor fixed points) is n . Thus the index of the geodesics are all multiples of $n - 1$ and if $n \geq 3$ the lacunary principle shows that the energy function on the loop space is a perfect Morse function. This easily implies that the Betti numbers grow exponentially. The same argument applies to arbitrarily many connected sums of $\mathbb{S}^n \times \mathbb{S}^n$ and hence provides a geometric proof of the well known fact that such manifolds are rationally hyperbolic.

REFERENCES

- [AA1] A.V. Alekseevsky and D.V. Alekseevsky, *G-manifolds with one dimensional orbit space*, Ad. in Sov. Math. **8** (1992), 1–31.
- [AA2] A.V. Alekseevsky and D.V. Alekseevsky, *Asystatic G-manifolds*, The Erwin Schroedinger International Institute for Mathematical Physics, Preprint.
- [A] M. Alexandrino, *On Polar Foliations and the Fundamental Group*, Results. Math. **60** (2011), 213–223.
- [AT] M. Alexandrino, and D. Töben, *Singular Riemannian foliations on simply connected spaces*, Differential Geom. Appl. **24** (2006), 383–397.
- [AW] S. Aloff & N. Wallach, *An infinite family of 7-manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. **81**(1975), 93–97.
- [BH] A. Back and W.Y. Hsiang, *Equivariant geometry and Kervaire spheres*, Trans. Amer. Math. Soc. **304** (1987), no. 1, 207–227.
- [Be] M. Berger, *Les varietes riemanniennes homogenes normales simplement connexes a Courbure strictment positive*, Ann. Scuola Norm. Sup. Pisa **15** (1961), 191–240.
- [BB] L. Bérard Bergery, *Les variétés riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive*, J. Math. Pures et Appl. **55** (1976), 47–67.
- [Ber] I. Bergman, *Reducible polar representations*, Manuscripta Math. **104** (2001), 309–324.
- [BD] J. Berndt and J. C. Daz-Ramos, *Polar actions on the complex hyperbolic plane*, to appear in AGAG.
- [BDT] J. Berndt, J.C. Daz-Ramos and H. Tamaru, *Hyperpolar homogeneous foliations on symmetric spaces of noncompact type*, J. Diff. Geom. **86** (2010), 191–235.
- [Bö] C. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, Invent. Math. **134** (1998), 145–176.
- [BWZ] C. Böhm, M. Wang, M. and W. Ziller, *A variational approach for compact homogeneous Einstein manifolds*, Geom. Funct. Anal. **14** (2004), 681–733.
- [Bo] H. Boualem, *Feuilletages riemanniennes singuliers transversalement integrables*, Compositio Math. **95**, 101–125.
- [Br] G.E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, Vol. 46. (1972), Academic Press, New York.
- [Ch] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Diff. Geom. **8** (1973), 623–628.
- [Cs] R. Cleyton and A. Swann, *Cohomogeneity one G_2 -structures*, J. Geom. Phys. **44**:202 (2002).
- [Co] D. Conti, *Cohomogeneity one Einstein-Sasaki 5-manifolds*, Commun. Math. Phys. **274** (2007), 751–774.
- [CGLP] M. Cvetič, G.W. Gibbons, H Lu and C.N. Pope, *New Cohomogeneity One Metrics With Spin(7) Holonomy*, J. Geom. Phys. **49** (2004) 350–365.
- [Da] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), 125–137.
- [Dav] M.W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series 32. Princeton University Press, Princeton, NJ, 2008.
- [De] O. Dearnicott, *A 7-manifold with positive curvature*, Duke Math. J. **158** (2011), 307–346.
- [EH1] J. Eschenburg and E. Heintze, *Polar representations and symmetric spaces*, J. Reine u. Ang. Math. **507** (1999), 93–106.
- [EH2] J. Eschenburg and E. Heintze, *On the classification of polar representations*, Math. Z. **232** (1999), 391–398.
- [FGT] F. Fang, K. Grove and G. Thorbergsson, *Tits Geometry and Positive Curvature*, arXiv:1205.6222.
- [FHT] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Springer Verlag 2001.
- [GM] F. Galaz-Garcia and M. Kerin, *Cohomogeneity two torus actions on non-negatively curved manifolds of low dimensions*, arXiv:1111.1640.

- [Gr] K. Grove, *Developments around positive sectional curvature*, Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, 117133, Surv. Differ. Geom., 13, Int. Press, Somerville, MA, 2009.
- [GH1] K. Grove and S. Halperin, *Contributions of rational homotopy theory to global problems in geometry*, Publ. Math. I.H.E.S. **56** (1982), 171–177.
- [GH2] K. Grove and S. Halperin, *Dupin hypersurfaces, group actions and the double mapping cylinder*, J. Differential Geom. **26** (1987), 429–459.
- [GS2] K. Grove, C. Searle, *Global G-manifold Reductions and Resolutions*, Ann. Global Anal. Geom. **18** (2000), 437–446.
- [GVZ] K. Grove, L. Verdiani and W. Ziller, *An exotic T_1S^4 with positive curvature*, Geom. Funct. Anal. **21** (2011), 499–524.
- [GWZ] K. Grove, B. Wilking and W. Ziller, *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry*, J. Diff. Geom. **78** (2008), 33–111.
- [GZ1] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. **152** (2000), 331–367.
- [GZ2] K. Grove and W. Ziller, *Lifting group actions and nonnegative curvature*, Trans. Amer. Math. Soc. **363** (2011) 2865–2890.
- [HH] I. Hambleton and J.C. Hausmann, *Equivariant bundles and isotropy representations*, Groups, Geometry and Dynamics **4** (2010), 127–162.
- [Ha] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.
- [HLO] E. Heintze, X. Lu, and C. Olmos, *Isoparametric submanifolds and a Chevalley-type restriction theorem*, Integrable systems, geometry, and topology, 151–190, AMS/IP Stud. Adv. Math., 36, Amer. Math. Soc., Providence, RI, 2006.
- [HPTT1] E. Heintze, R. Palais, C.-L. Terng, and G. Thorbergsson, *Hyperpolar actions on symmetric spaces*, Geometry, topology and physics for Raoul Bott, ed. S.-T. Yau, International Press, 1994, 214–245.
- [HPTT2] E. Heintze, R. Palais, C.-L. Terng, and G. Thorbergsson, *Hyperpolar actions and k-flat homogeneous spaces*, J. Reine Angew. Math. **454** (1994), 163–179.
- [Ho] C. Hoelscher, *Cohomogeneity one manifolds in low dimensions*, Pacific J. Math. **246** (2010), 129–185.
- [HHS] W.-T. Hsiang, W.-Y. Hsiang and I. Sterling, *On the construction of codimension two minimal immersions of exotic spheres into Euclidean spheres*, Invent. Math. **82** (1985), 447–460.
- [HL] W.Y. Hsiang and H.B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Differential Geometry **5** (1971), 1–38.
- [Ko1] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), 571–612.
- [Ko2] A. Kollross, *Polar actions on symmetric spaces*, J. Differential Geom. **77** (2007), 425–482.
- [Ko3] A. Kollross, *Low Cohomogeneity and Polar Actions on Exceptional Compact Lie Groups*, Transformation Groups, **14** (2009), 387–415.
- [KL] A. Kollross and A. Lytchak, *Polar actions on symmetric spaces of higher rank*, Preprint 2011.
- [La] J. Lauret, *Einstein solvmanifolds and nilsolitons*, Contemp. Math. **491** (2009), 1–35.
- [Lo] O. Loos, *Symmetric spaces*, Vol II, W.A. Benjamin, New York, (1969).
- [L] A. Lytchak, *Polar foliations on symmetric space*. Preprint 2012.
- [LT] A. Lytchak and G. Thorbergsson *Curvature explosion in quotients and applications*, J. Differential Geom. **85** (2010).
- [Me] R. Mendes, *Equivariant tensors on polar manifolds*, Preprint 2012.
- [Mu] M. Mucha, *Polar actions on certain principal bundles over symmetric spaces of compact type*, Proc. Amer. Math. Soc. **139** (2011), 2249–2255.
- [Oh1] H.S. Oh, *Toral actions on 5-manifolds*, Trans. Amer. Math. Soc. **278** (1983), 233–252.
- [Oh2] H.S. Oh, *6-dimensional manifolds with effective T^4 actions*, Topology Appl. **13** (1982), 137–154.
- [OR] P. Orlik and F. Raymond, *Actions of the torus on 4-manifolds I*, Trans. Amer. Math. Soc. **152** (1970), 531–559.
- [PT1] R. Palais and C.-L. Terng, *A general theory of canonical forms*, Trans. Amer. Math. Soc. **300** (1987), 771–789.
- [PT2] R. Palais and C.-L. Terng, *Critical point theory and submanifold geometry*, Lecture Notes in Mathematics, vol. 1353, Springer-Verlag, Berlin, Heidelberg, New York, 1988.
- [PP] G.P. Paternain and J. Petean, *Minimal entropy and collapsing with curvature bounded from below*, Inv. Math. **151** (2003), 415–450.
- [PTh] F. Podest and G. Thorbergsson, *Polar actions on rank-one symmetric spaces*, J. Differential Geom., **53** (1999), 131–175.

- [Ro] M. Ronan, *Lectures on buildings*, Perspectives in Mathematics **7**, Academic Press, Inc., Boston, MA, 1989.
- [Sz1] J. Szenthe, *Orthogonally transversal submanifolds and the generalizations of the Weyl group*, Period. Math. Hung. **15** (1984), 281–299.
- [Sz2] J. Szenthe, *A generalization of the Weyl group*, Acta Math. Hungar. **41** (1983), 347–357.
- [Ti2] J. Tits, *A local approach to buildings. The geometric vein*. The Coxeter Festschrift. Edited by C. Davis, B. Grünbaum, and F. A. Sherk, 519–547, Springer, New York-Berlin, 1981.
- [V1] L. Verdiani, *Cohomogeneity one Riemannian manifolds of even dimension with strictly positive sectional curvature, I*, Math. Z. **241** (2002), 329–339.
- [V2] L. Verdiani, *Cohomogeneity one manifolds of even dimension with strictly positive sectional curvature*, J. Differential Geom. **68** (2004), 31–72.
- [VZ] L. Verdiani and W. Ziller, *Concavity and rigidity in non-negative curvature*, Preprint 2012.
- [Wa] N. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. of Math. **96** (1972), 277–295.
- [Wan] M. Wang, *Einstein metrics from symmetry and bundle constructions*, Surveys in differential geometry: essays on Einstein manifolds, 287–325, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999.
- [Wi] B. Wilking, *A duality theorem for Riemannian foliations in nonnegative sectional curvature*, Geom. Funct. Anal. **17** (2007), 1297–1320.

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